TRACKING WITH BALANCE

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Abstract. This paper describes a tracking controller for a class of multi-input multi-output underactuated mechanical systems in which the unactuated degrees of freedom are acted upon and made unstable by gravitational forces. The tracking controller described allows approximate tracking of reference trajectories for the actuated degrees of freedom, while bounding the response of the unactuated degrees of freedom, thereby achieving output tracking with balance.

Keywords. Underactuated, Nonminimum Phase, Balance, Nonlinear Control, Internal Equilibrium, Mechanical Systems, Robot Control, Output Regulation

1. INTRODUCTION

This paper describes an approach to tracking control for multi-input multi-output underactuated nonlinear mechanical control systems having unstable zero-dynamics. We will be primarily concerned with systems in which we wish to cause actuated degrees of freedom to follow desired trajectories, but in which gravitational forces that act upon the unactuated degrees of freedom result in internal instability. In all such systems, accuracy of tracking must be sacrificed in order to maintain internal stability (balance). Examples of such systems are rockets and missiles, bicycles and motorcycles, trucks and automobiles which are tilted onto two side wheels, the classical inverted pendulum on a cart, and robotic manipulators with unstable payloads.

An intuitive feel for the problems with which this paper is concerned may be gained by considering the task of using a pencil to draw a large figure-eight on a horizontal sheet of paper, the pencil never leaving the paper, while at the same time balancing a broomstick on the back of your writing hand. Assume, furthermore, that the figure-eight is time-parameterized, i.e. it is desired that the pencil follow a particular $y_d(t) \in \mathbb{R}^2$ along the page, the path of $y_d(t)$ being the figure-eight. It is clear that if one ignores the configuration of the broomstick and draws the time-parameterized figure-eight as specified, the broom will fall. It is equally clear that one may pay attention to the state of the broomstick, moving one's hand to keep the broomstick balanced, but that the mark made by the pencil will not be the desired figure-eight. In a sense, this article shows how to draw the figure-eight (following $y_d(t)$) *approximately* while keeping the broomstick from falling over.

1.1 *Previous Work*

Though output tracking is a classical problem in automatic control, output tracking with internal stability continues to be a very active area of interest. Recent work in the area of trajectory generation for nonminimum phase systems appears in (Devasia *et al.*, 1994) and (Hunt *et al.*, 1994), the idea being that once a bounded state trajectory corresponding to exact output tracking is generated, that trajectory may then be stabilized through linearization and linear control methods (Devasia and Paden, 1994). The application of these methods necessitate presetting of initial conditions, a requirement we wish to avoid in the present paper. In (Benedetto and Lucibello, 1993) nonlinear tracking control is studied for the case where a known solution exists and where one is free to choose initial conditions. In (Grizzle *et al.*, 1994) it is shown that exact causal tracking of an open set of output reference trajectories, while maintaining internal stability, is impossible. We are, in fact, interested in tracking reference trajectories drawn from an open set, and we do want internal stability.

Virtually all work in tracking control for nonminimum phase systems adopts the philosophy of Francis (Francis, 1977) and (Isidori and Byrnes, 1990) in seeking a particular state solution corresponding to exact tracking, and then stabilizing that solution. In contrast, the method presented here and in (Getz, 1995*a*) stabilizes a nominal output tracking *error dynamics*. A submanifold $\mathcal{E}(t)$ of state-space is constructed, called an *internal equilibrium manifold*, and a neighborhood of $\mathcal{E}(t)$ is made attractive and invariant by choice of a control. Unlike the sliding surface in sliding mode control (see (Slotine and Li, 1991) for a review), no solution of the system resides in $\mathcal{E}(t)$ itself. But solutions in the neighborhood of the manifold approximately obey the nominal output tracking error dynamics.

The control method presented here is called *internal equilibrium control*. Internal equilibrium control of single-input, single-output systems has been described in detail in (Getz, 1995*a*). Applications have included tracking control for the inverted pendulum on a cart (Getz and Hedrick, 1995; Getz, 1995*a*), and tracking control for a bicycle (Getz and Marsden, 1995*a*; Getz, 1995*b*; Getz, 1995*a*). The present paper presents internal equilibrium control for multi-input, multi-output mechanical systems.

2. THE UNDERACTUATED SYSTEM

Consider an underactuated mechanical system with dynamic equations

$$\underbrace{\begin{bmatrix} M_{11}(q) & M_{12}(q) \\ M_{21}(q) & M_{22}(q) \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \phi_1(q,\dot{q}) \\ \phi_2(q,\dot{q}) \end{bmatrix}}_{\phi(q,\dot{q})} + \begin{bmatrix} \tau \\ 0 \end{bmatrix} \tag{1}$$

with $q_1 \in \mathbb{R}^m$, $q_2 \in \mathbb{R}^p$, $q = [q_1^T, q_2^T]^T \in \mathbb{R}^n$, $\tau \in \mathbb{R}^m$, $M_{11} \in \mathbb{R}^{m\times m}$, $M_{12} \in \mathbb{R}^{m\times p}$, $M_{21} \in \mathbb{R}^{p\times m}$, $M_{22} \in \mathbb{R}^{p\times p}$, $\phi_1 \in \mathbb{R}^m$, $\phi_1 \in \mathbb{R}^p$, with

$$\phi(q,\dot{q}) := \underbrace{\begin{bmatrix} C_{11}(q,\dot{q}) & C_{12}(q,\dot{q}) \\ C_{21}(q,\dot{q}) & C_{22}(q,\dot{q}) \end{bmatrix}}_{C(q,\dot{q})} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} F_1(q) \\ F_2(q) \end{bmatrix}}_{F(q)} \tag{2}$$

Assume that $p \leq m$, thus the number of unactuated degrees of freedom is no greater than the number of actuated degrees of freedom. The matrix $M(q) \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite inertia tensor, $F(q)$ is the vector of gravitational forces, and $C(q,\dot{q})\dot{q}$ is the centrifugal and Coriolis forces, homogeneously second order in $\dot{q}$, with $C_{11} \in \mathbb{R}^{m\times m}$, $C_{12} \in \mathbb{R}^{m\times p}$, $C_{21} \in \mathbb{R}^{p\times m}$, and $C_{22} \in \mathbb{R}^{p\times p}$. We will assume that $F(0) = 0$. Consequently $\phi(0,0) = 0$. Consider the **output** of system (1) to be $y = q_1$. Assume that all functions of q and $\dot{q}$ in (1) are sufficiently smooth.

Assumption 1. Assume that the origin of the linearization of (1) is exponentially stable. ▲

2.1 *External/Internal Convertible Form*

Define a new input u by the state-dependent transformation

$$\tau = M_{11}(q)u + M_{12}(q)\ddot{q}_2 - \phi_1(q,\dot{q}) \tag{3}$$

Using transformation (3) in (1) gives

$$\ddot{q}_1 = u \tag{4}$$

Substitute (4) into the last p equations of (1) to get

$$M_{22}(q)\ddot{q}_2 = \phi_2(q,\dot{q}) - M_{21}(q)u \tag{5}$$

Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $\alpha_1 = q_2$, $\alpha_2 = \dot{q}_2$, $x = [x_1^T, x_2^T]^T$, $\alpha = [\alpha_1^T, \alpha_2^T]^T$. In these new coordinates (1) takes the simple form of a first order system

$$\left\{ \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{\alpha}_1 &= \alpha_2 \\ M_{22}(x_1,\alpha_1)\dot{\alpha}_2 &= \phi_2(x,\alpha) + M_{21}(x_1,\alpha_1)u \end{aligned} \right. \tag{6}$$

with output $y = x_1$, and where we abuse notation by replacing the argument q by (x_1, α_1), and the argument $(q, \dot{q})$ by (x, α). Similarly we will use $F(x_1, \alpha_1)$ in place of $F(q_1)$ below.

Assumption 2. Assume $M_{21}(x_1,\alpha_1) \in \mathbb{R}^{p\times m}$ is full rank for all (x_1, α_1) in a neighborhood of $(0,0)$. ▲

Call x the **external state**, external because it is observable via the output y. Call α the **internal state**, internal because it is unobservable via the output y. The **external dynamics** of (6) are the x-dynamics with input u. The **internal dynamics** of (6) are the α-dynamics with input u, regarding x as an exogenous variable,

$$\left\{ \begin{aligned} \dot{\alpha}_1 &= \alpha_2 \\ M_{22}(x_1,\alpha_1)\dot{\alpha}_2 &= \phi_2(x,\alpha) + M_{21}(x_1,\alpha_1)u \end{aligned} \right. \tag{7}$$

Given two integers j and k with $0 < j < k$, and a C^k function $h(t) \in \mathbb{R}^s$, let

$$h^{(j,k)}(t) := [h^{(j)T}, h^{(j+1)T}, \ldots, h^{(k)T}](t)^T \subset \mathbb{R}^{s^{k-j+1}} \tag{8}$$

where $h^{(l)}(t)$ is the l^{th} derivative of $h(t)$ with respect to t, and where, by convention $h^{(0)}(t) = h(t)$. Assume that we have a C^2 reference trajectory $y_d(t) \in \mathbb{R}^m$ which, for brevity, we refer to as y_d. Disregarding the evolution of α, we may cause $y^{(0,1)} = x$ to converge exponentially to $y_d^{(0,1)} \in \mathbb{R}^{2m}$ as $t \to \infty$ by using the input

$$u_{\text{ext}}(y_d^{(0,2)}, x) = \ddot{y}_d - \gamma_2(x_2 - \dot{y}_d) - \gamma_1(x_1 - y_d) \tag{9}$$

where matrices $\gamma_i \in \mathbb{R}^{m\times m}$, $i \in \{1, 2\}$, are positive definite.

In the context of underactuated mechanical systems Spong (Spong, 1996) calls Assumption 2 *strong inertial coupling*. In a broader context, Assumption 2 makes (6) an *external/internal convertible system*, also called *E/I convertible* (see (Getz, 1995*a*), Chapter 6). The reason for this terminology may be seen as follows. For any matrix $Q \in \mathbb{R}^{p\times m}$, let $Q^{\mathrm{R}} \in \mathbb{R}^{m\times p}$ denote the **right inverse** of Q,

$$Q^{\mathrm{R}} := Q^T(QQ^T)^{-1} \tag{10}$$

Thus $QQ^{\mathrm{R}} = I \in \mathbb{R}^{p\times p}$. By a new choice of output $\lambda = \alpha_1$ and the input

$$u_{\text{int}}(v) = M_{21}^{\mathrm{R}}(x_1, \alpha_1)\left(-\phi(x, \alpha) + M_{22}(x_1, \alpha_1)v\right) \tag{11}$$

we may convert the internal state α to an external state, and the external state x to the internal state resulting in the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -M_{21}^{\mathrm{R}}(x_1, \alpha_1)\phi(x, \alpha) + M_{21}^{\mathrm{R}} M_{22}(x_1, \alpha_1)v \\ \dot{\alpha}_1 = \alpha_2 \\ \dot{\alpha}_2 = v \end{cases} \tag{12}$$

Given a C^2 reference trajectory $\lambda_d(t) \in \mathbb{R}^p$, by a choice of v,

$$v_{\text{int}}(\lambda_d^{(0,2)}, \alpha) = \ddot{\lambda}_d - \beta_2(\alpha_2 - \dot{\lambda}_d) - \beta_1(\alpha_1 - \lambda_d) \tag{13}$$

where matrices $\beta_i \in \mathbb{R}^{p\times p}$, $i \in \{1, 2\}$, are positive definite, we cause $\lambda^{(0,1)} = \alpha$ to converge exponentially to $\lambda_d^{(0,1)}$ as $t \to \infty$. Thus, under Assumption 2, the structure of (1) is such that we may cause y to track y_d, or λ to track λ_d. Referring to our introductory analogy of drawing a figure-eight, the input $u_{\text{ext}}(y_d^{(0,2)}, x)$ (9) corresponds to ignoring the state of the broomstick while drawing the figure-eight, and the input $u_{\text{int}}(v_{\text{int}}(\lambda_d^{(0,2)}, \alpha))$ (11), (13) (with $y_d \equiv 0$) corresponds to balancing the broomstick while ignoring the path of the pencil.

2.2 *Internal Instability*

The **zero dynamics** of (6) are the dynamics remaining under the restriction $y \equiv 0$ and $u \equiv 0$, namely

$$\begin{cases} \dot{\alpha}_1 = \alpha_2 \\ M_{22}(0, \alpha)\dot{\alpha}_2 = C_{22}(0, \alpha)\alpha_2 + F_2(0, \alpha_1) \end{cases} \tag{14}$$

If the zero dynamics are unstable at $\alpha = 0$, then (6) is said to be a *nonminimum phase* system.

Assumption 3. Assume that $\alpha = 0$ is an unstable equilibrium of (14). ▲

Since $C_{22}(0, \alpha)\alpha_2$ is homogeneously second order in α_2 and $F(0, 0) = 0$, and since we have assumed that (6) is linearizable at the origin, instability of the zero dynamics at the origin can only arise as a result of the gravitational forces $F_2(0, \alpha_1)$.

3. THE INTERNAL EQUILIBRIUM ANGLE

Our approach to controlling underactuated mechanical E/I convertible systems will be to construct a bounded function $\alpha_{\mathrm{e}} \in \mathbb{R}^p$ of x and $y_d^{(0,2)}$ such that if α_1 tracks α_{e}, then y approximately tracks y_d. Let

$$\begin{bmatrix} e_x^1 \\ e_x^2 \end{bmatrix} := \begin{bmatrix} x_1 - y_d \\ x_2 - \dot{y}_d \end{bmatrix}, \quad \begin{bmatrix} e_\alpha^1 \\ e_\alpha^2 \end{bmatrix} := \begin{bmatrix} \alpha_1 - \alpha_{\mathrm{e}} \\ \alpha_2 - \dot{\alpha}_{\mathrm{e}} \end{bmatrix} \tag{15}$$

with $e_x = [e_x^{1T}, e_x^{2T}]^T$ and $e_\alpha = [e_\alpha^{1T}, e_\alpha^{2T}]^T$. Referring back to the introductory analogy, the configuration of the broomstick may be specified with two angles α_1^1 and α_1^2. Using a control of the form $u_{\text{int}}(v_{\text{int}})$ (see (11) and (13)) we can cause $\alpha_1 = [\alpha_1^1, \alpha_1^2]^T$ to converge to any desired angular trajectory. We want to find angles α_{e}^1 and α_{e}^2 such that if α_1 tracks $\alpha_{\mathrm{e}} = [\alpha_{\mathrm{e}}^1, \alpha_{\mathrm{e}}^2]^T$ approximately, then the pencil converges to the figure-eight path approximately. In order to correct errors in the position and velocity of the pencil, α_{e} must depend on the desired position and velocity of the pencil, as well as the actual position and velocity of the pencil. The angles α_{e} must also correspond to an *upright* broomstick.

The function $\alpha_{\mathrm{e}}(y_d^{(0,2)}, x)$ is defined implicitly as follows. In the internal dynamics equation (7) set $u = u_{\text{ext}}$, $\dot{\alpha}_1 = 0$, and $\dot{\alpha}_2 = 0$ to create a set of $2p$ nonlinear equations

$$\begin{cases} 0 = \alpha_2 \\ 0 = C_{21}(x, \alpha)x_2 + C_{22}(x, \alpha)\alpha_2 + F_2(x_1, \alpha_1) \\ \qquad + M_{21}(x_1, \alpha_1)u_{\text{ext}}(y_d^{(0,2)}, x) \end{cases} \tag{16}$$

Since all functions in (16) are assumed to be smooth, and since the origin is an equilibrium of (6), a solution α exists for $y_d^{(0,2)}$ and x sufficiently close to the origin.

The α_2 part of the solution is, by the first p equations, just $\alpha_2 = 0$. We call the α_1 part of the equations $\alpha_{\rm e}$. Define

$$\begin{aligned}\psi(\alpha_1, y, x) := {} & C_{21}(x; \alpha_1, 0)x_2 + F_2(x_1, \alpha_1) \\ & + M_{21}(x_1, \alpha_1) u_{\rm ext}(y, x)\end{aligned} \tag{17}$$

Since equations (16) dictate that $\alpha_2 = 0$, with respect to determining the solution $\alpha_{\rm e}(y_d^{(0,2)}, x)$, equations (16) are equivalent to

$$\psi(\alpha_{\rm e}, y_d^{(0,2)}, x) = 0 \tag{18}$$

Under the following assumption, and through the implicit function theorem, equations (18) define $\alpha_{\rm e}$ implicitly as a function of $y_d^{(0,2)}$ and x.

Assumption 4. For all y_d satisfying $\sup_{t\geq 0} \|y_d(t)\|_\infty < \epsilon$, and $\|x\|$ sufficiently small,

$$\frac{\partial}{\partial \alpha_1} \psi(\alpha_{\rm e}, y_d^{(0,2)}(t), x) \neq 0 \tag{19}$$

▲

We call $\alpha_{\rm e}$ the **internal equilibrium angle** (Getz, 1995*a*) of (16). From $\alpha_{\rm e}$ we may form a t-dependent *graph* over the $2m$-dimensional subspace of the state-space of (16), the subspace in which x resides. We call this graph the **internal equilibrium manifold** $\mathcal{E}(t)$, defined as

$$\mathcal{E}(t) = \{(x, \alpha) \in \mathbb{R}^n \mid \alpha_1 = \alpha_{\rm e}(y_d^{(0,2)}(t), x), \alpha_2 = 0\} \tag{20}$$

Assumption 5. Assume that $\alpha_{\rm e}(y_d^{(0,2)}, x)$ is C^4 in its arguments, with all mixed partial derivatives bounded for $\|y_d^{(0,2)}\|_\infty < \epsilon$ and $\|x\| < r$. ▲

For brevity let $u_{\rm ext}$ denote $u_{\rm ext}(y_d^{(0,2)}, x)$, and let

$$\begin{array}{ll} \bar{M}_{21} := M_{21}(x_1, \alpha_{\rm e}) & \bar{F}_2 := F_2(x_1, \alpha_3) \\ \bar{C}_{21} := C_{21}(x; \alpha_{\rm e}, 0) & \bar{A}_2 := A_2(x; \alpha_{\rm e}, 0) \end{array} \tag{21}$$

so that the defining equation for $\alpha_{\rm e}$ becomes

$$0 = \bar{C}_{21} x_2 + \bar{F}_2 - \bar{M}_{21} u_{\rm ext} \tag{22}$$

The input space $\mathbb{R}^m$ may be decomposed into two orthogonal subspaces consisting of the range-space of $\bar{M}_{21}^T$, and the null-space, or kernel, of $\bar{M}_{21}$. Let $u_{\rm ext}^{\rm r}$ denote that part of $u_{\rm ext}$ in the range space of $\bar{M}_{21}^T$, and let $u_{\rm ext}^{\rm n}$ denote that component of $u_{\rm ext}$ in the null-space of $\bar{M}_{21}$. The orthogonal components of $u_{\rm ext}$ may be constructed from $\bar{M}_{21}$ and $u_{\rm ext}$ by

$$u_{\rm ext}^{\rm r} = \bar{M}_{21}^{\rm R} \bar{M}_{21} u_{\rm ext} \qquad u_{\rm ext}^{\rm n} = (I - M_{21}^{\rm R} \bar{M}_{21}) u_{\rm ext} \tag{23}$$

It is clear from (23) that

$$u_{\rm ext}^{\rm r} + u_{\rm ext}^{\rm n} = u_{\rm ext} \tag{24}$$

Now note that it follows from (22) that

$$\begin{aligned} u_{\rm ext}^r &= \bar{M}_{21}^{\rm R} (\bar{C}_{21} x_2 + \bar{F}_2) \\ &= M_{21}(x, \alpha_{\rm e})^{\rm R} \left(C_{21}(x; \alpha_{\rm e}, 0) x_2 + F_2(x_1, \alpha_{\rm e})\right) \end{aligned} \tag{25}$$

Below in Section 5.1 we will use the right-hand side of (25) as the first term in a Taylor expansion about $(x; \alpha_{\rm e}, 0)$.

Remark 6. The internal equilibrium angle $\alpha_{\rm e}$ is the solution to an implicit nonlinear equation. To produce an explicit and arbitrarily accurate estimate of α_e we may use a dynamic inverter (Getz and Marsden, 1995*b*; Getz, 1995*a*), a nonlinear dynamic system constructed from (22) which tracks the solution α_e. ▲

4. ESTIMATING THE DERIVATIVES OF THE INTERNAL EQUILIBRIUM ANGLE

Define the **nominal external vector field** $N_{\rm ext} \in \mathbb{R}^{2m}$ as

$$N_{\rm ext} := \begin{bmatrix} x_2 \\ u_{\rm ext}(y_d^{(0,2)}(t), x) \end{bmatrix} \tag{26}$$

If we could ignore the internal dynamics of (6), the vector field $N_{\rm ext}$ would cause $y^{(0,1)} = x$ to converge to $y_d^{(0,1)}(t)$ exponentially as $t \to \infty$. Since it is our intention to cause x to track $y_d^{(0,1)}(t)$ approximately, if x is indeed close to $y_d^{(0,1)}(t)$, then $\dot{x}$ is close to $N_{\rm ext}$. The first and second Lie derivatives of $\alpha_{\rm e}$ along the vector field $N_{\rm ext}$ are

$$\begin{aligned} L_{N_{\rm ext}} \alpha_{\rm e}(y_d^{(0,3)}(t), x) := {} & D_1 \alpha_{\rm e}(y_d^{(0,2)}(t), x) \cdot y_d^{(1,3)}(t) \\ & + D_2 \alpha_{\rm e}(y_d^{(0,2)}(t), x) \cdot N_{\rm ext} \end{aligned} \tag{27}$$

and

$$\begin{aligned} L_{N_{\rm ext}}^2 \alpha_{\rm e}(y_d^{(0,4)}(t), x) := {} & D_1 L_{N_{\rm ext}} \alpha_{\rm e}(y_d^{(0,3)}(t), x) \cdot y_d^{(1,4)}(t) \\ & + D_2 L_{N_{\rm ext}} \alpha_{\rm e}(y_d^{(0,3)}(t), x) \cdot N_{\rm ext} \end{aligned} \tag{28}$$

Thus if x is close to $y_d^{(0,1)}(t)$, then $L_{N_{\rm ext}} \alpha_{\rm e}$ is close to $\dot{\alpha}_{\rm e}$, and $L_{N_{\rm ext}}^2 \alpha_{\rm e}$ is close to $\ddot{\alpha}_{\rm e}$. We will use $L_{N_{\rm ext}} \alpha_{\rm e}$ and $L_{N_{\rm ext}}^2 \alpha_{\rm e}$ to approximate the first and second time derivatives of $\alpha_{\rm e}$ in our control law. Since the actual values of $\dot{\alpha}_{\rm e}$ and $\ddot{\alpha}_{\rm e}$ depend upon the control law itself, their use in the control law would result in a dynamic controller. Our approximations will allow us to use a non-dynamic controller.

5. THE INTERNAL EQUILIBRIUM CONTROLLER

Let

$$\begin{cases} u_{\mathrm{e}}(v_{\mathrm{e}}) := M_{21}^{\mathrm{R}}(x_1, \alpha_1)\left(-\phi_2(x,\alpha) + M_{22}(x,\alpha)v_{\mathrm{e}}\right) \\ \qquad + \left(I - \bar{M}_{21}^{\mathrm{R}}\bar{M}_{21}\right) u_{\mathrm{ext}}(y_d^{(0,2)}, x) \\ v_{\mathrm{e}} := L_{N_{\mathrm{ext}}}^2 \alpha_{\mathrm{e}} - \beta_2(\alpha_2 - L_{N_{\mathrm{ext}}}\alpha_{\mathrm{e}}) \\ \qquad -\beta_1(\alpha_1 - \alpha_{\mathrm{e}}) \end{cases} \tag{29}$$

Assuming that the value of α_{e} is accessible (see Remark 6), the equations (29) constitute a controller for (6). Under conditions defined below in Section (5.1), $u_{\mathrm{e}}(v_{\mathrm{e}})$ causes $e = [e_x^T, e_\alpha^T]^T$ to be uniformly ultimately bounded.

If e is uniformly ultimately bounded, then x converges uniformly to an invariant neighborhood of $y_d^{(0,2)}(t)$, and (x, α) converges uniformly to an invariant neighborhood of $\mathcal{E}(t)$ as $t \to \infty$.

5.1 *Analysis of the Controller*

Substituting $u_{\mathrm{e}}(v_{\mathrm{e}})$ for u in (6) gives

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -M_{21}^{\mathrm{R}}\phi_2(x,\alpha) + (I - M_{21}^{\mathrm{R}}M_{21})u_{\mathrm{ext}} \\ \qquad +M_{21}^{\mathrm{R}}M_{22}v_{\mathrm{e}} \\ \dot{\alpha}_1 = \alpha_2 \\ \dot{\alpha}_2 = v_{\mathrm{e}} \end{cases} \tag{30}$$

Expand v_{e} as

$$\begin{aligned} v_{\mathrm{e}} &= L_{N_{\mathrm{ext}}}^2\alpha_{\mathrm{e}} - \beta_1(\alpha_2 - L_{N_{\mathrm{ext}}}\alpha_{\mathrm{e}}) - \beta_1(\alpha_1 - \alpha_{\mathrm{e}}) \\ &= \ddot{\alpha}_{\mathrm{e}} - \beta_2(\alpha_2 - \dot{\alpha}_{\mathrm{e}}) - \beta_1(\alpha_1 - \alpha_{\mathrm{e}}) \\ &\quad +(L_{N_{\mathrm{ext}}}^2\alpha_{\mathrm{e}} - \ddot{\alpha}_{\mathrm{e}}) + \beta_2(L_{N_{\mathrm{ext}}}\alpha_{\mathrm{e}} - \dot{\alpha}_{\mathrm{e}}) \\ &= \ddot{\alpha}_{\mathrm{e}} - \beta_2 e_\alpha^2 - \beta_1 e_\alpha^1 + p_\alpha(y_d^{(0,4)}, e) \end{aligned} \tag{31}$$

where

$$\begin{aligned} p_\alpha(y_d^{(0,4)}, e) &:= (L_{N_{ext}}^2\alpha_{\mathrm{e}} - \ddot{\alpha}_{\mathrm{e}}) + \beta_2(L_{N_{\mathrm{ext}}}\alpha_{\mathrm{e}} - \dot{\alpha}_{\mathrm{e}}) \\ &= O(\|y_d^{(0,4)}\|, \|e\|) \end{aligned} \tag{32}$$

Also expand u_{e} in a Taylor series about $(x; \alpha_{\mathrm{e}}, 0)$ as

$$\begin{aligned} u_{\mathrm{e}}(v_{\mathrm{e}}) &= -M_{21}^{\mathrm{R}}(x,\alpha)\phi_2(x,\alpha) + (I - M_{21}^{\mathrm{R}}M_{21})u_{\mathrm{ext}} \\ &\quad +M_{21}^{\mathrm{R}}M_{22}v_{\mathrm{e}} \\ &= -\bar{M}_{21}^{\mathrm{R}}(\bar{C}_{21}x_2 + \bar{F}_2) + (I - \bar{M}_{21}^{\mathrm{R}}\bar{M}_{21})u_{\mathrm{ext}} \\ &\quad +p_x(y_d^{(0,4)}, e) \\ &= u_{\mathrm{ext}}^r + u_{\mathrm{ext}}^n + p_x(y_d^{(0,4)}, e) \\ &= u_{\mathrm{ext}} + p_x(y_d^{(0,4)}, e) \end{aligned} \tag{33}$$

where $p_x(y_d^{(0,4)}, e) = O(\|y_d^{(0,4)}\|, \|e\|)$.

Substitute these expansions into (30) to get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y}_d - [\gamma_1|\gamma_2] \cdot e_x + p_x(y_d^{(0,4)}, e) \\ \dot{\alpha}_1 = \alpha_2 \\ \dot{\alpha}_2 = \ddot{\alpha}_{\mathrm{e}} - [\beta_1|\beta_2] \cdot e_\alpha + p_\alpha(y_d^{(0,4)}, e) \end{cases} \tag{34}$$

Transform the system (34) into e_x and e_α coordinates to get

$$\begin{cases} \dot{e}_x^1 = e_x^2 \\ \dot{e}_x^2 = -[\gamma_1|\gamma_2] \cdot e_x + p_x(y_d^{(0,4)}, e) \\ \dot{e}_\alpha^1 = e_\alpha^2 \\ \dot{e}_\alpha^2 = -[\beta_1|\beta_2] \cdot e_\alpha + p_\alpha(y_d^{0,4)}, e) \end{cases} \tag{35}$$

We may regard (35) as an exponentially stable error system

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_\alpha \end{bmatrix} = \begin{bmatrix} A_x & 0 \\ 0 & A_\alpha \end{bmatrix} \begin{bmatrix} e_x \\ e_\alpha \end{bmatrix} \tag{36}$$

perturbed by a term

$$p(y_d^{(0,4)}, e) := \begin{bmatrix} 0 \\ p_x(y_d^{(0,4)}, e) \\ 0 \\ p_\alpha(y_d^{(0,4)}, e) \end{bmatrix} \tag{37}$$

where

$$A_x = \begin{bmatrix} 0 & I \\ -\gamma_1 & -\gamma_2 \end{bmatrix}, \quad A_\alpha = \begin{bmatrix} 0 & I \\ -\beta_1 & -\beta_2 \end{bmatrix} \tag{38}$$

Assume that $\|y_d^{(0,4)}\|_\infty < \epsilon$ for some $\epsilon > 0$. Then for some real numbers $k_1 > 0$, $k_2 > 0$, and $r > 0$, with $\|e\| < r$,

$$\|p_\alpha(y_d^{(0,4)}, e)\| \leq k_1\epsilon + k_2\|e\| \tag{39}$$

An additive perturbation such as $p(y_d^{(0,4)}, e)$ satisfying (39) is called an *affine* perturbation.

Assumption 7. Assume that γ_1, γ_2, β_1, and β_2 have been chosen so that A_x and A_α are Hurwitz, and the linearization of (35) at the origin is stable. ▲

For any integer $k > 0$, and real number $r > 0$, let $\mathcal{B}_r \subset \mathbb{R}^k$ denote the open ball about the origin $\{\chi \in \mathbb{R}^k \mid \|\chi\| < r\}$.

Proposition 8. (Getz, 1995*a*) **Convergence for the Internal Equilibrium Controller.** Assume

$$\sup_{t\geq 0} \|y_d^{(0,n)}(t)\|_\infty < \epsilon \tag{40}$$

for some $\epsilon \geq 0$. Assume also that $\|(e_x(0), e_\alpha(0))\| \leq r_e$ for some $r_e > 0$, where $(e_x(t), e_\alpha(t))$ is the solution of (35). If r_e and k_2 (see (39)) are sufficiently small real numbers, then there exists a $t_1 \geq 0$, and a class-$\mathcal{K}$ function $b(\epsilon)$ such that for all $(e_x(0), e_\alpha(0)) \in \mathcal{B}_{r_e} \subset \mathbb{R}^n$, $(e_x(t), e_\alpha(t))$ converges toward zero exponentially with a rate $\gamma > 0$ until $(e_x(t_1), e_\alpha(t_1))$ enters $\mathcal{B}_{b(\epsilon)}$. Once $(e_x(t), e_\alpha(t))$ enters $\mathcal{B}_{b(\epsilon)}$ it remains in $\mathcal{B}_{b(\epsilon)}$ thereafter. See Figure 1 ◆

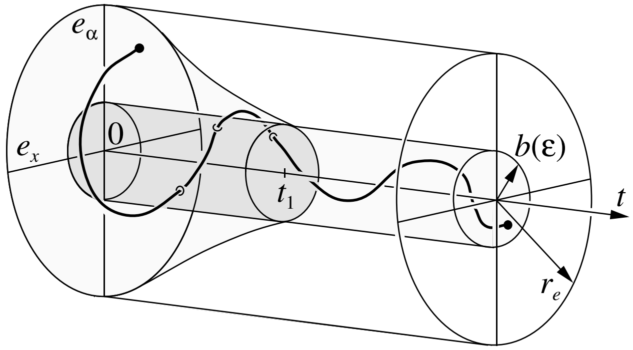

Fig. 1. Ultimately bounded error convergence. See Proposition 8.

Remark 9. A consequence of Proposition 8 is that the tracking error $y(t) - y_d(t)$ is uniformly ultimately bounded as is the error $\alpha_1 - \alpha_e$. ▲

The proof of Proposition 8, given in (Getz, 1995*a*) (Proposition 6.7.4), relies on Theorem 6.7.5 of (Getz, 1995*a*) regarding the ultimately bounded convergence of exponentially stable systems subject to affine perturbations. Theorem 6.7.5 of (Getz, 1995*a*) is, in turn, a variation of classical results on perturbed systems (see (Khalil, 1992), Section 4.5).

6. CONCLUSIONS

Using the internal equilibrium controller requires that y_d be C^n because $\alpha_e^{(p)}$ depends on $y_d^{(n)}$. There is nothing intrinsic to the form of our controller which allows us to guarantee that a sufficiently small k_2 (see (39)) exists for each problem one might encounter. If necessary one could estimate an upper bound on k_2 for a given ϵ and r_e using a computer.

Even when $y_d^{(0,m-1)} \equiv 0$, there are still conditions on k_2 required in order to assure exponential stability. By choosing gain matrices β_1, β_2, γ_1, and γ_2 such that (35) has a stable linearization, however, we know that for an arbitrarily small region about the origin, the origin in exponentially stable. The gain matrices may be chosen so that the linearization of the system at the origin is, for instance, a linear quadratic regulator. So in the worst case, we have good regulation in a neighborhood of the origin irrespective of k_2. More usually, however, the tracking performance is substantially enhanced in comparison to linear methods. For a single-input single-output comparison of a linear quadratic regulator to an internal equilibrium controller having an identical linearization to the linear regulator, see (Getz, 1995*a*), Chapter 6. Future work will illustrate the performance benefits on specific applications.

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