DYNAMIC INVERSION OF NONLINEAR MAPS

Neil H. Getz and Jerrold E. Marsden

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CENTER FOR PURE AND APPLIED MATHEMATICS

Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720
Dynamic Inversion of Nonlinear Maps*

Neil H. Getz
Department of Electrical Engineering and Computer Sciences
University of California at Berkeley
Berkeley, CA 94720
getz@eeecs.berkeley.edu

Jerrold E. Marsden
Control and Dynamical Systems 104-44
California Institute of Technology
Pasadena, CA 91125
marsden@cds.caltech.edu

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Abstract

We present a method for constructing dynamic systems, called “dynamic inverters”, which solve inverse problems having time-varying vector-valued solutions. We introduce the notion of a dynamic inverse of a map and use the dynamic inverse in the construction of the dynamic inverter. Our results generalize and extend previous results on the inversion of maps using continuous-time dynamic systems. By posing the dynamic inverse itself as the solution to an inverse problem, we show how one may track a dynamic inverse dynamically while simultaneously using the dynamic inverse to solve for the time-varying root of interest. Dynamic inversion is a continuous-time analog computational paradigm that may be incorporated into controllers in order to continuously provide estimates of time-varying parameters necessary for control. This allows nonlinear control systems to be posed entirely in continuous-time, replacing discrete root-finding algorithms as well as discrete algorithms for matrix inversion with integration. Example applications include solving for the intersection of time-varying polynomials, as well as inversion of nonlinear control systems.

Keywords: dynamical systems, dynamic inversion, matrix inversion, neural networks, homotopy, gradient flow, optimization, inverse problems.

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1 Introduction

In this paper we describe a continuous-time dynamic methodology for inverting nonlinear maps. We call this methodology dynamic inversion. Given a map $F : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ known, by some means, to have a continuous isolated solution $\theta_*(t)$ to $F(\theta, t) = 0$, we associate with $F(\theta, t)$ another map $G[w, \theta, t]$ which we call a dynamic inverse of $F(\theta, t)$. The map $G[w, \theta]$ is characterized by the property that the dynamic system

$$\dot{\theta} = -G[F(\theta, t), \theta, t]$$

has a solution $\theta(t)$ which converges to the solution $\theta_*(t)$.

1.1 An Informal Introduction to Dynamic Inversion

Dynamic inversion is most easily introduced\(^2\) by first considering the problem of finding the root of a real-valued function on the real line.

A. Consider the function $F : \mathbb{R} \rightarrow \mathbb{R}; \theta \mapsto F(\theta)$ illustrated in Figure 1. Assume that we do not know the solution $\theta_*$ to $F(\theta) = 0$, but that we would like to find it using a representation of $F(\theta)$. The representation of $F(\theta)$ may be in the form of a closed-form expression, a combination of table-lookup and interpolation, a physical (non-dynamic) system with an input $\theta$ and an output $F(\theta)$, or any combination of the above. Assume that we know that a unique solution exists in the interval $[a, b] \subseteq \mathbb{R}$. The function $F(\theta)$ of Figure 1 has a number of features which limit the choices of techniques that may be used to find its root, $\theta_*$. It is, in places, not differentiable. It also has minima and maxima at points

\(^1\)We will use the terms “map,” “mapping”, and “function” interchangeably.

\(^2\)The precise hypotheses will be developed starting in Section 2 below.
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other than \( \theta_* \). We may even be uncertain about the value of \( F(\theta) \) for \( \theta \) in certain regions of \([a, b]\) as indicated by the shaded region of the graph\(^3\). We will assume, however, that \( F(\theta) \) is continuous on \([a, b]\). Clearly Newton’s method and its variants (e.g. secant method, regula falsi) would fail to find the root of this function if the initial guess at the root is not close to \( \theta_* \). However, we make the following claim:

Claim 1.1 For any initial value \( \theta_0 \in [a, b] \), the solution \( \theta(t) \) to the dynamic system

\[
\dot{\theta} = -F(\theta)
\]

converges to the root \( \theta_* \) as \( t \to \infty \). \( \topICA \)

Informal Proof of Claim 1.1: Consider a solution of (1.2). Assume that \( F(\theta) \) is such that a solution \( \theta(t) \) of (1.2) exists for any \( \theta(0) \in [a, b] \). If \( \theta(0) = \theta_* \), then \( F(\theta(0)) = 0 \), so (1.2) works fine for this case. If \( \theta(0) \in [a, \theta_*] \), then the vector field \(-F(\theta)\) pushes the state to the right, towards \( \theta_* \). As long as \( F(\theta(t)) < 0 \) this will continue to be so. Since \( F(\theta) < 0 \) for all \( \theta \in [a, \theta_*] \), the solution \( \theta(t) \) will flow to \( \theta_* \). Likewise, if \( \theta(0) \in [\theta_*, b] \), then \(-F(\theta)\) pushes the solution \( \theta(t) \) left towards \( \theta(t) \) until \( \theta(t) = \theta_* \). \( \topICA \)

The argument above suggests that for maps similar to \( F(\theta) \) in Figure 1, i.e. maps whose values are strictly above the abscissa to the right of \( \theta_* \) and strictly below the abscissa to the left of \( \theta_* \), \( \theta(t) \to \theta_* \) asymptotically as \( t \to \infty \). We will make an additional claim, however.

Claim 1.2 The convergence \( \theta \to \theta_* \) where \( \theta(t) \) is the solution of (1.2) is in fact exponential, that is, there exists a \( k_1 \) and a \( k_2 \) in \( \mathbb{R} \), \( 0 < k_i < \infty, \ i \in \{1, 2\} \) such that for all \( t > 0 \),

\[
||\theta(t)|| \leq k_1 e^{-k_2 t}
\]

\( \topICA \)

The important feature of \( F(\theta) \) in Figure 1 which allows us to make Claim 1.2 is illustrated in Figure 2. Note that to the right of the root \( \theta_* \), the graph of \( F(\theta, t) \) is above a line of slope \( \beta \) passing through \((\theta_*, 0)\), and to the left of \( \theta_* \), the graph of \( F(\theta, t) \) is below the same line. An equivalent expression of this

\( ^3 \)We assume that there exists some \( k > 0 \) such that \( F(\theta) \leq k < 0 \) in the region of Figure 1 marked by “?”. 

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Figure 2: There exists a line passing through \((\theta_*, 0)\) of slope \(\beta > 0\) such that \(F(\theta)\) (shown gray) is above the line to the right of \(\theta_*\) and below the line to the left of \(\theta_*\).

feature is to define \(z := \theta - \theta_*\) and say that for all \(z \in [a - \theta_*, b - \theta_*]\),

\[
z F(z + \theta_*) \geq \beta z^2 \tag{1.4}
\]

**Informal Proof of Claim 1.2:** Let \(V(\theta) := \frac{1}{2}(\theta - \theta_*)^2 = \frac{1}{2}z^2\). Differentiate \(V(\theta)\) with respect to \(t\) to get

\[
\frac{d}{dt} V(\theta) = (\theta - \theta_*) \dot{\theta} = -z F(\theta) \tag{1.5}
\]

But from (1.4) we have

\[-z F(\theta) = -z F(z + \theta_*) \leq -\beta z^2 \tag{1.6}\]

Note that

\[
\beta z^2 = 2\beta \frac{1}{2} z^2 = 2\beta V(\theta) \tag{1.7}
\]

Therefore

\[
V(\theta) \leq V(\theta(0)) e^{-2\beta t} \tag{1.8}
\]

Insert the definition of \(V(\theta)\) into (1.8) to get

\[
\frac{1}{2} (\theta(t) - \theta_*)^2 \leq \frac{1}{2} (\theta(0) - \theta_*)^2 e^{-2\beta t} \tag{1.9}
\]
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Figure 3: Any function $F(\theta)$ that is transverse to the $\theta$-axis at $\theta_*$ and whose values lay in the shaded regions of either of these graphs may be inverted with the dynamic system (1.13).

Multiply (1.9) by 2 and take the positive branch of the square root of both sides of the resulting equation to get

$$|\theta(t) - \theta_*| \leq |\theta(0) - \theta_*| e^{-\beta t}, \quad \text{for } t \geq 0$$

(1.10)

which proves the claimed exponential convergence.

We call the dynamic system (1.2) a **dynamic inverter** for $\theta_*$ since it solves $F(\theta_*) = 0$ for $\theta_*$.

**B.** Let $\text{sign}(a)$ be defined by

$$\text{sign}(a) = \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \end{cases}$$

(1.11)

If we replace (1.4) by

$$z \text{ sign}(F(b) - F(a)) \cdot F(z + \theta_*) \geq \beta z^2$$

(1.12)

then the dynamic inverter

$$\dot{\theta} = -\text{sign}(F(b) - F(a)) \cdot F(\theta)$$

(1.13)

will suffice for inversion of functions of same form as $F(\theta)$ in Figure 1 or for functions such as $-F(\theta)$.

Consider Figure 3. Any function $F(\theta)$ which is transverse to the $\theta$-axis and whose values lie in the gray regions of the figure will be dynamically inverted by (1.13), as long as $F(\theta)$ is such that a solution of (1.13) exists for all $\theta(0) \in [a, b]$. The proof is similar to the proof above, the essential step in the proof coming from the inequality (1.12).
Figure 4: The function $(\theta - 1)^3$ with root $\theta_* = 1$. No line of slope $\beta$ may be drawn through $\theta_*$ as in Figure 2.

C. Now suppose that we encounter a function

$$F(\theta) := (\theta - c)^3$$

which is graphed in Figure 4 for $c = 1$. This function has a well defined root $\theta_* = c$, but there does not exist a $\beta > 0$ such that (1.12) holds, i.e. there is no line of constant slope $\beta > 0$ such that $F(\theta)$ fits in either picture of Figure 3. However, consider the following observation:

**Observation 1.3** If $G : \mathbb{R} \to \mathbb{R}; \ w \mapsto G[w]$, is such that

$$G[w] = 0 \implies w = 0$$

then

$$\{G[w] = 0 \text{ and } F(\theta_*) = 0\} \implies G[F(\theta_*)] = 0$$

In other words, if $\theta_*$ is a root of $F(\theta)$, then $\theta_*$ is also a root of $G[F(\theta)]$.

Observation 1.3 affords us the freedom to generalize our dynamic root solving method to functions such as (1.14). For instance, let $G[w] = \text{sign}(w)|w|^{1/3}$. Then $G[F(\theta)] = \theta - c$ which satisfies (1.12) for any $\beta \in (0,1]$. So for dynamic inversion of (1.14) we could use

$$\dot{\theta} = -G[F(\theta)]$$
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Figure 5: The function \( G[w] := \text{sign}(w)|w|^{1/4} \), a dynamic inverse of \( F(\theta) = (\theta - 1)^3 \).

But note that we could also use \( G[w] := \text{sign}(w)|w|^{1/4} \), shown in Figure 5. The composition \( G[F(\theta)] \) is shown in Figure 6 along with an appropriate line of slope \( \beta = 1/2 \). In fact there are an infinite number of functions \( G[w] \) which satisfy

\[
zG[F(z + \theta_*)] \geq \beta z^2
\]

for some \( \beta > 0 \). We call such a function \( G[w] \) a dynamic inverse of \( F(\theta) \) since, in the context of the dynamic system (1.17), \( G[w] \) solves \( F(\theta) = 0 \) with exponential convergence of \( \theta(t) \to \theta_* \).

D. The bisection method (see, for instance, [1], page 84) could also be used to solve for the root of \( F(\theta) \), but the bisection method relies upon the fact that \( \theta_* \) divides any continuous interval containing \( \theta_* \) into two connected sub-intervals, one in which \( F(\theta) \) is positive, and the other in which \( F(\theta) \) is negative. The bisection method is defined only for real-valued functions of one variable. On the other hand, we will see that dynamic inversion, including the criterion (1.18), generalizes easily to maps \( F: \mathbb{R}^n \to \mathbb{R}^n \) as well as maps \( F(\theta, t) \) which depend on time.

1.2 Previous Work

Continuous-time dynamic methods of solving inverse problems have been around a long time. Indeed, one could cast any dynamic system \( \dot{x} = \phi(x, t) \) with an
Figure 6: The composition $G[F(\theta)] = \text{sign}((\theta - 1)^3)|(\theta - 1)^3|^{1/4}$. Now we can draw a line of slope $\beta = 1/2$ (dashed) through $(\theta_*, 0) = (1, 0)$, like the line in Figure 2. The dotted curve is $F(\theta) = (\theta = 1)^3$.

asymptotically stable isolated equilibrium $x_*$ as a dynamic inverter that solves for its own equilibrium point, the equilibrium point being the solution to the inverse problem of finding $x_*$ that satisfies

$$\phi(x_*, t) = 0$$

(1.19)

In the areas of adaptive control [2, 3] and optimal control [4, 5] dynamical systems have been used to solve for unknown parameters of physical systems. Most results are for linear systems whose parameters are assumed to be slowly varying. Such results may be used also for nonlinear systems that can be converted to linear systems through state-dependent coordinate transformations [3].

With the recent vogue of designing dynamic systems and circuits thought to mimic certain models of computation in the nervous system [6, 7], there has been a renewed interest in viewing dynamics as computation. Gradient flows, in particular, have been heavily relied upon in the neural network literature [8]. More recently Brockett [9, 10] has shown how continuous-time dynamical systems may be used to sort lists and solve linear programming problems. Helmke and Moore in [11] review a broad variety of inverse and optimization problems solvable by continuous-time dynamical systems.

The inverse-kinematics problem in which one wishes to solve for $\theta_*(t)$
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satisfying

\[ \mathcal{F}(\theta) = x_d(t) \]  \hspace{1cm} (1.20)

has given rise to a number of related continuous-time dynamic methods [12, 13, 14, 15, 16, 17, 18] of solving inverse problems of the form (1.20). The notion of a dynamic inverse of a nonlinear map, introduced here, generalizes the role of \( D\mathcal{F}(\theta)^{-1} \cdot \dot{w} \) and \( D\mathcal{F}(\theta)^T \cdot \dot{w} \) in the dynamic inverters of [13, 14, 15, 16, 17, 18]. Our definition will allow us to use dynamic inversion itself to determine a dynamic inverse, while simultaneously using that dynamic inverse to solve for a time-varying root. Also, we have developed dynamic inversion around the inversion of maps of a more general form \( F(\theta, t) \) than \( \mathcal{F}(\theta) - x_d(t) \). In [19, 20] we apply our methods to the inverse-kinematics problem, in particular to the problem of controlling a robotic arm to track an inverse-kinematic solution. In the present paper, however, we will give example applications (see for instance Examples 4.7 and 4.8) of dynamic inversion to the solution of inverse problems which are not of the form (1.20), and are thus not solvable by any other continuous-time dynamic techniques we have found. Also, unlike previous methods, differentiability of \( F(\theta, t) \) is not required.

1.3 Main Results

The main results of this paper are as follows:

1. We define a dynamic inverse for nonlinear maps.

2. Using a dynamic inverse we construct a dynamic system that yields an estimate of the root \( \theta_*(t) \) of \( F(\theta, t) = 0 \), and we prove that the estimation error is bounded as \( t \to \infty \).

3. We construct a derivative estimator for the root \( \theta_*(t) \), and incorporate that estimator into a dynamic system which estimates \( \dot{\theta}_*(t) \) with vanishing error as \( t \to \infty \).

4. We construct a dynamic system that dynamically solves for a dynamic inverse as the solution to an inverse problem, while simultaneously using that dynamic inverse to produce an estimator for a root \( \theta_*(t) \), where the estimation error is vanishing.

1.4 Overview

In Section 2 we will introduce the formal definition of a dynamic inverse of a map. Then in Section 3 we will use the dynamic inverse to derive a continuous-time dynamic estimator for time-varying vector-valued roots of nonlinear time-dependent maps. We will prove two theorems which assert that the resulting estimation error may be made arbitrarily small within an arbitrarily short period of time by the adjustment of a single scalar gain. With one theorem we will assert that the estimation error becomes arbitrarily small in finite time; with
the other theorem we will assert exponential convergence of an estimator to the root as \( t \to \infty \). We will then show in Section 4 how a dynamic inverse itself may be determined dynamically, that is we will pose both the dynamic inverse itself and the root we seek as the solution to an equation of the form \( F(\theta, t) = 0 \). In Section 5 we will discuss the generalization of dynamic inversion. Examples will be used throughout this paper to illustrate important features of the definitions and theorems.

A number of simple examples will illustrate application of dynamic inversion in cases where closed-form solutions are readily available, allowing the reader to verify the theory and operation of dynamic inversion. In Example 4.7, however, we apply dynamic inversion to solve for the intersection of two time-varying polynomials, a problem whose quasiperiodic solution is not so readily available in a closed form. Then, in a final example, Example 4.8, we apply dynamic inversion to the inversion of time-varying nonlinear control systems in order to create a dynamic controller that gives the control system a desired dynamics.

2 A Dynamic Inverse

We begin by defining the dynamic inverse, a definition central to the development of the methodology presented in this paper. The dynamic inverse is defined in terms of the unknown root of a map. Later we will show that a dynamic inverse may be obtained without first knowing the root.

**Definition 2.1** For \( F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n; (\theta, t) \mapsto F(\theta, t) \) let \( \theta^*(t) \) be a continuous isolated solution of \( F(\theta, t) = 0 \). A map \( G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n; (w, \theta, t) \mapsto G[w, \theta, t] \) is called a dynamic inverse of \( F(\theta, t) \) on the ball \( \mathcal{B}_r := \{ z \in \mathbb{R}^n \mid \| z \| \leq r \}, r > 0 \), if

1. the map \( G[F(\theta, t), \theta, t] \) is Lipschitz in \( \theta \), piecewise-continuous in \( t \), and

2. there is a real constant \( \beta \), with \( 0 < \beta < \infty \), such that

\[
\text{Dynamic Inverse Criterion} \quad z^T G[F(z + \theta^*(t), t), z + \theta^*(t), t] \geq \beta \| z \|_2^2 \tag{2.1}
\]

for all \( z \in \mathcal{B}_r \).

In order to emphasize the association of a particular parameter \( \beta \) with \( G \), we will sometimes say that \( G \) is a dynamic inverse with parameter \( \beta \). We will also, at times, restrict the domain of \( t \) to some subset of \( \mathbb{R}_+ \).

Note that the definition of dynamic inverse does not involve dynamics, though it’s significance will be in the dynamic context of a dynamic inverter as will be seen in Section 3.
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Some easily verified properties of the dynamic inverse that prove useful are the following:

**Property 2.2 Positive Scalar Times Dynamic Inverse.** If \( G[w, \theta, t] \) is a dynamic inverse of \( F(\theta, t) \) with parameter \( \beta \), then for any \( \mu > 0 \in \mathbb{R}, \mu G[w, \theta, t] \) is a dynamic inverse of \( F(\theta, t) \) with parameter \( \mu \beta \). ▲

**Property 2.3 Many \( \beta \)'s for Each Dynamic Inverse.** If \( G[w, \theta, t] \) is a dynamic inverse of \( F(\theta, t) \) with parameter \( \beta_1 \), then for any \( \beta_2 \) such that \( 0 < \beta_2 \leq \beta_1 \), \( G[w, \theta, t] \) is a dynamic inverse of \( F(\theta, t) \) with parameter \( \beta_2 \). ▲

**Property 2.4 Stacking Decoupled Dynamic Inverses.** Assume that \( G_1(w_1, \theta_1, t) \) is a dynamic inverse of \( F_1(\theta_1, t) \) with parameter \( \beta_1 \), and \( G_2(w_2, \theta_2, t) \) is a dynamic inverse of \( F_2(\theta_2, t) \) with parameter \( \beta_2 \). Let \( w = (w_1, w_2) \) and \( \theta = (\theta_1, \theta_2) \). Let \( G \) and \( F \) be defined by

\[
G[w, \theta, t] := \begin{bmatrix} G_1(w_1, \theta_1, t) \\ G_2(w_2, \theta_2, t) \end{bmatrix}, \quad F(\theta, t) := \begin{bmatrix} F_1(\theta_1, t) \\ F_2(\theta_2, t) \end{bmatrix}
\]  

(2.2)

Then \( G \) is a dynamic inverse of \( F \) with parameter \( \beta = \min\{\beta_1, \beta_2\} \). ▲

**Property 2.5 Dynamic Inverse at Zero.** Let \( \bar{F}(z, t) := F(z + \theta_*(t), t) \) and \( \bar{G}[z, \theta, t] := G[w, z + \theta_*(t), t] \). Then \( G[w, \theta, t] \) is a dynamic inverse of \( F(\theta, t) \) relative to a solution \( \theta_* \) if and only if \( G[w, z, t] \) is a dynamic inverse of \( \bar{F}(z, t) \) relative to \( \theta_* = 0 \). ▲

**Property 2.6 Trivial Dynamic Inverse.** If \( G_1(w, \theta, t) \) is a dynamic inverse of \( F_1(\theta, t) \), then \( G_2(w) = w \) is a dynamic inverse of \( G_1(F_1(\theta, t), \theta, t) \). When \( G[w] = kw \) where \( k \in \mathbb{R}, 0 < k < \infty \), then \( G[w] \) is called a trivial dynamic inverse. ▲

It will be proven in the next section that if \( F(\theta, t) \) has a dynamic inverse \( G[w, \theta, t] \), then for all initial conditions \( \theta(0) \) in an open neighborhood of \( \theta_*(0) \), the integral curves of the vector field \( -\mu G[F(\theta, t), \theta, t] \) converge exponentially to a neighborhood of \( \theta_*(t) \) as \( t \to \infty \).

For the case of a scalar valued \( F(\theta, t) \) we have the following lemma:

**Lemma 2.7 Dynamic Inverse for Scalar Functions.** Let \( F: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}; (\theta, t) \mapsto F(\theta, t) \) be \( C^2 \) in \( \theta \) and continuous in \( t \) for all \( \theta \) in an interval \([a, b]\). Let \( \theta_*(t) \) be a continuous isolated solution of \( F(\theta, t) = 0 \). Assume that there exists an \( r > 0 \) and a \( \beta > 0 \) such that

\[
(\theta - \theta_*) \text{sign}(F(b, t) - F(a, t)) F(\theta, t) \geq \beta(\theta - \theta_*)^2
\]  

(2.3)
for all \((\theta - \theta_*) \in \mathcal{B}_r\) and all \(t \in \mathbb{R}_+\). Then

\[
\text{Dynamic Inverse for Scalar Functions}
\]

\[
G[w] := \text{sign} \left( \frac{\partial}{\partial \theta} D_1 F(\theta_*(0), 0) \right) \cdot w
\]

(2.4)

is a constant dynamic inverse of \(F(\theta, t)\).

\[\diamondsuit\]

**Remark 2.8** Lemma 2.7 tells us that for time-varying scalar valued \(C^1\) functions, we need only pick a sign to produce a dynamic inverse. Typically one knows an interval \([a, b]\) that brackets the solution. Then one need only evaluate \(F(a, t_1)\) and \(F(b, t_2)\) for any times \(t_1 \geq 0\) and \(t_2 \geq 0\) to determine a dynamic inverse. \[\blacktriangleleft\]

**Proof of Lemma 2.7:** Since \(F(\theta, t)\) is \(C^1\) in \(\theta\), \(D_1 F(\theta, t)\) is well-defined and continuous in \(\theta\) and \(t\). Since \(F(\theta, t)\) is continuous and satisfies (2.3) for all \(t \in \mathbb{R}_+\), \(\text{sign}(F(b, t) - F(a, t))\) is well-defined and constant for all \(t\) and, furthermore,

\[
\text{sign} \left( \frac{\partial}{\partial \theta} D_1 F(\theta_*(t), t) \right) = \text{sign}(F(b, t) - F(a, t))
\]

(2.5)

Therefore

\[
\text{sign} \left( \frac{\partial}{\partial \theta} D_1 F(\theta_*(t), t) \right) = \text{sign} \left( \frac{\partial}{\partial \theta} D_1 F(\theta_*(0), 0) \right)
\]

(2.6)

Thus the sign of \(D_1 F(\theta_*(t), t)\) is an invariant of the isolated solution \(\theta_*(t)\). Now from (2.3) and (2.5) we have

\[
(\theta - \theta_*)\text{sign} \left( \frac{\partial}{\partial \theta} D_1 F(\theta_*(0), 0) \right) F(\theta, t) \geq \beta(\theta - \theta_*)^2
\]

(2.7)

so \(G[w]\) (2.4) is a constant dynamic inverse for \(F(\theta, t)\).

\[\blacksquare\]

Dynamic inverses for affine maps are easily obtained as illustrated by the following example.

**Example 2.9 Dynamic Inverse for Affine Maps.** Let

\[
F(\theta, t) = A(\theta - u(t))
\]

(2.8)

where \(A \in \mathbb{R}^{n \times n}\). Then for any matrix \(B \in \mathbb{R}^{n \times n}\) such that \(BA\) is positive-definite, \(G[w, \theta, t] = B \cdot w\) is a dynamic inverse of \(F\). The solution \(\theta_*\) of \(F(\theta, t) = 0\) is \(\theta_*(t) = u(t)\). It is clear that

\[
z^T G [F(z + u(t), \theta, t), t] = z^T B(Az) \geq \sigma_{\min}(BA) \|z\|^2
\]

(2.9)
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where $\sigma_{\text{min}}(BA)$ is the smallest singular-value of $BA$. Note that if $A$ is singular, then $F$ given by (2.8) has no dynamic inverse. If $A$ is non-singular, a possible choice of $B$ is $A^T$. ▲

We will have occasion in Section 4 below to choose a dynamic inverse for one inverse problem to depend on the solution to a different, but related inverse problem. In such cases the combination of the two inverse problems may be viewed as a single inverse problem through the following property of dynamic inverses.

**Property 2.10 Stacking Coupled Dynamic Inverses.**
Assume that $G^1(w^1; \theta^1, \theta^2; t)$ is a dynamic inverse of $F^1(\theta^1, t)$, for all $\theta^2$ such that $(\theta^2 - \theta^2_0) \in B_{r_2}$, and $G^2(w^2; \theta^2, \theta^1; t)$ is a dynamic inverse of $F^2(\theta^2, t)$ for all $\theta^1$ such that $(\theta^1 - \theta^1_0) \in B_{r_1}$. Let $\theta := (\theta^1, \theta^2)$ and $w = (w^1, w^2)$. Then

$$G[w, \theta, t] := \begin{bmatrix} G^1(w^1; \theta^1, \theta^2; t) \\ G^2(w^2; \theta^2, \theta^1; t) \end{bmatrix}$$ (2.10)

is a dynamic inverse of

$$F(\theta, t) := \begin{bmatrix} F^1(\theta^1, t) \\ F^2(\theta^2, t) \end{bmatrix}$$ (2.11)

for all $(||\theta^1 - \theta^1_0||, ||\theta^2 - \theta^2_0||) \in B_{r_1} \times B_{r_2}$. ▲

Sufficient conditions on $F$ under which a dynamic inverse exists are mild. They are given in the following existence lemma.

**Lemma 2.11 Sufficient Conditions for Existence of a Dynamic Inverse.** For $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(\theta, t) \mapsto F(\theta, t)$, let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$. Let $F(\theta, t)$ be $C^2$ in $\theta$ and continuous in $t$. Assume that the following are true:

1. $D_1 F(\theta_*(t), t)$ is nonsingular for all $t$;
2. $D_1 F(\theta_*(t), t)$ and $D_1 F(\theta_*(t), t)^{-1}$ are bounded uniformly in $t$;
3. for all $z \in B_r$, $D_1^2 F(z + \theta_*(t), t)$ is bounded uniformly in $t$.

Under these conditions there exists an $r > 0$ independent of $t$, and a function $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$, $(w, \theta, t) \mapsto G[w, \theta, t]$ such that for each $t > 0$ and for all $\theta$ satisfying $\theta - \theta_*(t) \in B_r$, $G[w, \theta, t]$ is a dynamic inverse of $F(\theta, t)$. ▲

**Proof of Lemma 2.11:** Let

$$\tilde{F}(z, t) := F(z + \theta_*(t))$$ (2.12)

Since $D_1 F(\theta_*, t)$ is invertible for all $t \in \mathbb{R}_+$, by the inverse function theorem (see [21], Theorem 2.5.7, page 121), for each $t \in \mathbb{R}_+$ there exists an open
neighborhood $\mathcal{N}_t \subset \mathbb{R}^n$ of the origin, and a function $\tilde{F}^{-1} : \mathbb{R}^n \times [w, t] \rightarrow \mathbb{R}^n$; $(w, t) \mapsto \tilde{F}^{-1}(w, t)$ such that for all $z \in \mathcal{N}_t$,
\[
\tilde{F}^{-1} \left[ \tilde{F}(z, t), t \right] = z
\]
(2.13)

Let
\[
G[w, \theta, t] := \tilde{F}^{-1}(w, t)
\]
(2.14)

If there exists an $r > 0$ such that
\[
\mathcal{B}_r \subset \mathcal{N}_t \quad \forall t \in \mathbb{R}^n
\]
(2.15)

then for all $z \in \mathcal{B}_r$
\[
z^T G \left[ F \left( z + \theta_\ast(t), t \right), \theta, t \right] = z^T z = \|z\|_2^2
\]
(2.16)
and we may choose $G$ as a dynamic inverse with $\beta$ satisfying $0 < \beta \leq 1$. In the absence of items 2 and 3 of the hypothesis, there is the possibility that no such $r$ exists, e.g. the largest ball contained in $\mathcal{N}_t$ may be $\mathcal{B}_0$ in the limit as $t \to \infty$. Assurance that an $r > 0$ exists is provided by a proposition of Abraham, Marsden, and Ratiu [21] (Proposition 2.5.6, page 119) regarding size of the ball on which $\tilde{F}(z, t) = 0$ is solvable. Though that proposition gives explicit bounds on $r$ based on the explicit uniform bounds on $D_1 F(\theta_\ast(t), t), D_1 F(\theta_\ast(t), t)^{-1}$, and $D_2^2 F(z + \theta_\ast(t), t)$, for our purposes it is enough to know that the existence of such uniform bounds is sufficient to guarantee the existence of an $r > 0$.

Though Lemma 2.11 requires $F(\theta, t)$ to be $C^2$ in $\theta$ at $\theta = \theta_\ast(t)$, this is only a sufficient condition for the existence of a dynamic inverse. That it is not necessary is indicated by the next example.

**Example 2.12** Consider the piecewise-linear time-varying function
\[
F(\theta, t) = \begin{cases} 
-(\theta - u(t)), & \theta - u(t) \geq 0 \\
-\frac{1}{2}(\theta - u(t)), & \theta - u(t) < 0
\end{cases}
\]
(2.17)

where $u(t)$ is a continuous function of $t$. The solution to $F(\theta, t) = 0$ is $\theta_\ast(t) = u(t)$. Let\footnote{Throughout we will use the abuse of notation demonstrated by referring to $G[w, \theta, t]$ as $G[w]$ when the value of $G$ depends only on $w$.} $G[w, \theta, t] = G[w] = -w$. Then
\[
z^T G \left[ F \left( z + \theta_\ast(t) \right) \right] = -z^T F(z + u(t))
\]
\[
= \begin{cases} 
-z^T (-1), & z \geq 0 \\
-z^T (-1/2), & z < 0
\end{cases}
\]
\[
\geq \frac{1}{2} \|z\|^2
\]
(2.18)

where $\tilde{F}$ is as defined in (2.12), so that $0 < \beta \leq 1/2$. But $F(\cdot, t)$ is not differentiable at $\theta = \theta_\ast(t)$.
Dynamic Inversion of Nonlinear Maps

Using the exact inverse of $F$, namely (2.14) as a dynamic inverse as in the proof of Lemma 2.11 is not very practical since the exact inverse, though always a dynamic inverse, is normally not known. There is reason for hope, however, in the observation that the criterion that $G$ be a dynamic inverse of $F$ is considerably weaker than the criterion that $G$ be an inverse of $F$ in the usual sense. One might guess that a truncated Taylor expansion for $F^{-1}$ would be a good candidate for $G$. That this guess is true is verified in the proof of the following theorem.

**Theorem 2.13** Fixed Jacobian Inverse as a Dynamic Inverse. Let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$, where $F(\theta, t)$ is $C^2$ in $\theta$, and $C^1$ in $t$. Let $\bar{F}(z, t) := F(\theta_*(t) + z, t)$. Assume that $D_1 \bar{F}(0, t)$ is nonsingular, and that $D_1^2 \bar{F}(0, t)$ is bounded. Let $t_1 > 0$ be a constant. Then there exist $t_0$ and $t_2$, with $0 \leq t_0 < t_1 < t_2$, and an $r(t_1) \in \mathbb{R}$, $r(t_1) > 0$, such that for any $y \in B_{\gamma(t_1)}$,

$$G[w] = D_1 \bar{F}(y, t_1)^{-1} \cdot w$$  \hspace{1cm} (2.19)

is a dynamic inverse of $\bar{F}(z, t)$ for all $z \in B_{\gamma(t_1)}$ and all $t \in (t_0, t_2)$.

**Remark 2.14** Theorem 2.13 tells us that over a sufficiently small time interval, there is an open set of constant matrices such that if $M$ is in that set, then $G[w] := M \cdot w$ is a dynamic inverse of $F(\theta, t)$. See Figure 7.
Figure 7: For any $y \in B_{r(t_1)}$, the constant matrix $D_1 \tilde{F}(y, t_1)^{-1}$ provides a
dynamic inverse for $F(\theta, t)$ over a sufficiently small interval $(t_0, t_2)$ containing
$t_1$. See Theorem 2.13.

Remark 2.15 Nearby Jacobian Inverse as a Dynamic Inverse. We may
replace $t_1$ by $t$ in (2.19) to conclude from Theorem 2.13 that $D_1 \tilde{F}(\theta(t), t)^{-1} \cdot w$
is a dynamic inverse of $F(\theta, t)$ for all $t$, if $\theta(t)$ is sufficiently close to $\theta_*(t)$ for all
t $\in \mathbb{R}_+$. This will prove particularly important later when we use the dynamic
inverse in a dynamic context in order to keep $\theta(t)$ close to $\theta_*(t)$.

Proof of Theorem 2.13: Note that since $\tilde{F}(0, t) = 0$ for all $t$, if $\tilde{F}(z, t)$ is $C^k$
in $z$, then $D_2 \tilde{F}(0, t_1) \equiv 0$ for $l \in k$. Using this, we expand $\tilde{F}(z, t)$ in a Taylor
series in both variables to get

$$\tilde{F}(z, t) = D_1 \tilde{F}(0, t_1) \cdot z + O (||z||^2, |t - t_1||z||)$$  (2.20)

For $r > 0$, let $y \in B_r \subset \mathbb{R}^n$ and expand $D_1 \tilde{F}(0, t_1)$ about $y$ as

$$D_1 \tilde{F}(0, t_1) = D_1 \tilde{F}(y, t_1) + O(||y||)$$  (2.21)

Substitute (2.21) into (2.20) to get

$$\tilde{F}(z, t) = D_1 \tilde{F}(y, t_1) \cdot z + f(z, t)$$  (2.22)

where

$$f(z, t) = O (||z||^2, |t - t_1||z||, ||y|| ||z||)$$  (2.23)

^5By $O(a_1, a_2, \ldots, a_k)$ we mean $O(a_1) + O(a_2) + \cdots + O(a_k)$. 

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Dynamic Inversion of Nonlinear Maps

Now consider the dynamic inverse candidate

\[ G[w] = D_1 \tilde{F}(y, t_1)^{-1} \cdot w \]  

(2.24)

Left multiply \( G[\tilde{F}(z, t)] \) by \( z^T \) and expand \( \tilde{F} \) according to (2.22) to get

\[ z^T G \left[ \tilde{F}(z, t) \right] = z^T D_1 \tilde{F}(y, t_1)^{-1} \tilde{F}(z, t) = z^T z + z^T D_1 \tilde{F}(y, t_1)^{-1} f(z, t) \]  

(2.25)

Choose \( \beta \in (0, 1) \). If there exists an \( r \in \mathbb{R}_+ \) and an interval \((t_0, t_2)\) containing \( t_1 \) such that for all \( z \in \mathcal{B}_r \) and all \( t \in (t_0, t_2) \),

\[ z^T D_1 \tilde{F}(y, t_1)^{-1} f(z, t) \geq (\beta - 1) ||z||^2 \]  

(2.26)

then

\[ z^T G[\tilde{F}(z, t)] \geq \beta ||z||^2 \]  

(2.27)

implying that \( G \) is a dynamic inverse of \( \tilde{F} \) on \( \mathcal{B}_r \) for \( t \in (t_0, t_2) \). Since \( f(z, t) \) satisfies (2.23),

\[ D_1 \tilde{F}(y, t_1)^{-1} \cdot f(z, t) = O \left( ||z||^2, ||t - t_1||, ||y||, ||z|| \right) \]  

(2.28)

Thus for each \( t_1 > 0 \) there is always a sufficiently small \( r(t_1) > 0 \), and a sufficiently small interval \((t_0, t_2)\) such that (2.26) is true for the chosen \( \beta \). ■

**Remark 2.16 Positive-Definite Combinations with the Jacobian Inverse.** Let \( F(\theta, t) \) be \( C^1 \) in \( \theta \). Then any matrix valued function \( B(\theta, t) \in \mathbb{R}^{n \times n} \) such that \( B \) is continuous in \( t \), and

\[ B(\theta_*, t) D_1 F(\theta_*, t) > 0 \]  

(2.29)

\( G[w, \theta, t] = B(\theta, t) \cdot w \) is a dynamic inverse of \( F(\theta, t) \) for all \( \theta \) sufficiently close to \( \theta_\ast \). This includes as special cases \( B(t) = D_1 F(\theta, t)^{-1} \) and \( B(t) = D_1 F(\theta, t)^T \), where \( ||\theta - \theta_*|| \) is sufficiently small. ▲

Though it will often be convenient to choose a linear dynamic inverse, a dynamic inverse need not be linear as shown by the following two examples.

**Example 2.17 Nonlinear Dynamic Inverse.** Let \( F(\theta, t) = (\theta - \sin(t))^3 \) so that \( \theta_\ast = \sin(t) \). Note that \( F(\theta, t) \) fails to satisfy the conditions of Lemma 2.11. Let \( G[w] := \text{sign}(w)|w|^{1/3} \). Then

\[ z^T G[F(z + \theta_\ast, t)] = z^T z \geq ||z||^2 \]  

(2.30)

so \( G[w] \) is a dynamic inverse of \( F(\theta, t) \). Note that, though \( G[w] \) itself is not Lipschitz in \( w \), \( G[F(\theta, t)] = \theta - \sin(t) \) which is Lipschitz in \( \theta \) and continuous in \( t \), thus \( G[w] \) satisfies items 1 and 2 of the dynamic inverse definition, Definition 2.1. ▲
Example 2.18 Dynamic Inverse from Taylor Series Reversion. Let
\[ F(\theta, t) = \tan(\theta - \sin(t)) \]  \hspace{1cm} (2.31)

We may obtain a dynamic inverse of \( F \) through Taylor series reversion\(^6\) of \( \tan(z) \).
Let \( G[w, \theta, t] = G[w] = w - w^3/3 \). It is easily verified graphically that \( G[w] \) is a dynamic inverse of \( F(\theta, t) \) for \( \beta = 1/4 \) and for all \( z \in \mathcal{B}_1 \). ▲

Later in Section 4 we will show how a dynamic inverse can be determined dynamically, that is, we will find both the root and the dynamic inverse itself using a single dynamic system.

3 Dynamic Inversion

In this section we will use the dynamic inverse to construct a dynamic system whose state is an estimator for the root \( \theta_*(t) \) of \( F(\theta, t) = 0 \). We present two theorems, Theorem 3.1 and Theorem 3.5, collectively called the dynamic inversion theorem, covering both the case where we have no estimate of \( \theta_*(t) \) as well as the case in which we do have such an estimate.

3.1 Dynamic Inversion with Bounded Error

Suppose we wish to produce an estimate for the root \( \theta_*(t) \) of a map \( F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \). Assume that we have a rough estimate of \( \theta_*(0) \), called \( \theta^0 \), with \( \theta^0 - \theta_*(0) \in \mathcal{B}_r \), but no estimator for \( \theta_* \). Theorem 3.1 below tells us that once we have found a dynamic inverse \( G[w, \theta, t] \) we are guaranteed that there always exists a real \( \mu > 0 \) such that the solution \( \theta(t) \) to
\[ \dot{\theta} = -\mu G[F(\theta, t), \theta, t], \quad \theta(0) = \theta^0 \]  \hspace{1cm} (3.1)
approaches \( \theta_*(t) \) arbitrarily closely in an arbitrarily short period of time.

The following theorem is quite general in that it allows us to find roots of continuous, but not necessarily differentiable nonlinear maps.

Theorem 3.1 Dynamic Inversion Theorem—Bounded Error. For \( F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \); \( (\theta, t) \to F(\theta, t) \), let \( \theta_*(t) \) be a continuous isolated solution of \( F(\theta, t) = 0 \) for all \( t \in \mathbb{R}_+ \). Let \( G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \); \( (w, \theta, t) \to G[w, \theta, t] \) be a dynamic inverse of \( F(\theta, t) \) with parameter \( \beta \) for all \( z \in \mathcal{B}_r \) and \( t \in \mathbb{R}_+ \). Let \( E : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \); \( (\theta, t) \to E(\theta, t) \), be Lipschitz in \( \theta \) and piecewise-continuous in \( t \). Assume that there exists a \( \gamma \in \mathbb{R}_+ \) such that
\[ \| E \left( z + \theta_*(t), t \right) - \dot{\theta}_*(t) \|_\infty \leq \gamma/2 \]  \hspace{1cm} (3.2)

\(^6\)One reverts a Taylor series, \( q(z) = a_0 + a_1 z + a_2 z^2 + O(|z|^3) \) by solving for the coefficients \( b_i \) of a polynomial \( p(z) = b_0 + b_1 z + b_2 z^2 \) under the constraint that \( p(q(z)) = z \).
for all $t \in \mathbb{R}_+$, and $z \in B_r$. Assume also that $\|\theta(0) - \theta_*(0)\|_\infty$ is in $B_r$. Then for each $\mu \in \mathbb{R}_+$ satisfying $\mu \geq \gamma/\beta r$, the solution $\theta(t)$ to

<table>
<thead>
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<th>Dynamic Inverter with Bounded Error</th>
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<tbody>
<tr>
<td>$\dot{\theta} = -\mu G [F(\theta, t), \theta, t] + E(\theta, t)$</td>
</tr>
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satisfies

$$
\begin{cases}
    \|z(t)\| \leq \|z(0)\| e^{-\mu \beta t/2}, & \text{for } 0 \leq t \leq t_1 \\
    \|z(t)\| \leq \gamma/\mu \beta, & \text{for } t > t_1
\end{cases}
$$

where $z := \theta - \theta_*$ and

$$
t_1 = \frac{2}{\mu \beta} \ln \left( \frac{\mu \beta \|z(0)\|_2}{\gamma} \right)
$$

Proof of Theorem 3.1: Let $z(t) := \theta(t) - \theta_*(t)$, and assume that $\mu \geq \gamma/\beta r$. Transforming (3.3) to $z$-coordinates and letting $\tilde{F}(z, t) := F(z + \theta_*(t), t)$ and $\tilde{G}[w, z, t] = G[w, z + \theta_*(t), t]$ gives

$$
\dot{z} = -\mu \tilde{G} \left[ \tilde{F}(z, t), z + \theta_*(t), t \right] + E(z + \theta_*(t), t) - \dot{\theta}_*(t)
$$

Since $G[F(\theta, t), \theta, t]$ and $E(\theta, t)$ are Lipschitz in $\theta$ and piecewise-continuous in $t$, a solution $z(t)$, $t \in \mathbb{R}_+$ exists for (3.3). Let

$$
V(z) = \frac{1}{2} \|z\|_2^2
$$

Then

$$
\dot{V}(z) = z^T \dot{z}
$$

$$
= -z^T \mu \tilde{G} \left[ \tilde{F}(z, t), z, t \right] + z^T \left( E(z + \theta_*(t), t) - \dot{\theta}_*(t) \right)
$$

By (3.2),

$$
z^T \left( E(z + \theta_*(t), t) - \dot{\theta}_*(t) \right) \leq z^T \frac{1}{2} \gamma \|z\|_2 = \frac{1}{2} \gamma \|z(t)\|_2
$$

Combining (3.8) and (3.9) along with the assumption that $G[w, \theta, t]$ is a dynamic inverse of $F(z, t)$ with parameter $\beta$, we have

$$
\dot{V}(z) \leq -\mu \beta \|z(t)\|_2^2 + \frac{1}{2} \gamma \|z(t)\|_2
$$

$$
= -\frac{1}{2} \mu \beta \|z(t)\|_2^2 - \frac{1}{2} \mu \beta \|z(t)\|_2^2 + \frac{1}{2} \gamma \|z(t)\|_2
$$

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Therefore, for \( z(t) \) satisfying \( \|z(t)\|_2 \geq \gamma/\mu \beta \),

\[
\dot{V}(z) \leq -\frac{1}{2} \mu \beta \|z(t)\|_2^2 = -\mu \beta V(z) \tag{3.11}
\]

Let \( y(t) \) satisfy \( y(0) = V(z(0)) \), and \( \dot{y} = -\mu \beta y \). Then

\[
y(t) = \frac{1}{2} \|z(0)\|_2^2 e^{-\mu \beta t} \tag{3.12}
\]

It follows that if \( \|z(t)\|_2 \geq \gamma/\mu \beta \), then \( V(z(t)) \leq y(t) \). As a consequence,

\[
\|z(t)\|_2 \leq \|z(0)\|_2 e^{-\mu \beta t/2} \tag{3.13}
\]

as long as \( \|z(t)\|_2 \geq \gamma/\mu \beta \). If \( z(0) \in \mathcal{B}_{\gamma/\mu \beta} \), then since \( \dot{V} \leq 0 \) on the boundary of \( \mathcal{B}_{\gamma/\mu \beta} \), \( z(t) \) can never leave \( \mathcal{B}_{\gamma/\mu \beta} \). If \( z(0) \notin \mathcal{B}_{\gamma/\mu \beta} \), then \( z(t) \) is guaranteed to enter \( \mathcal{B}_{\gamma/\mu \beta} \) no later than \( t_1 \), where \( t_1 \) is the solution to

\[
\frac{\gamma}{\mu \beta} = \|z(0)\|_2 e^{-\mu \beta t_1/2} \tag{3.14}
\]

namely (3.5).

The map \( E(\theta, t) \) in Theorem 3.1 may model an estimator for \( \dot{\theta}_* \). It may also model errors resulting from the representation of \( F \) or the presence of noise of various sorts.

**Remark 3.2** Note that differentiability of \( F(\theta, t) \) is not a requirement for application of Theorem 3.1.

**Example 3.3** Dynamic Inversion of a Piecewise Linear Map – No \( \dot{\theta}_* \) Estimate. Consider the map \( F : [-4, 4] \subset \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
F(\theta, t) = \sin(4\pi t) + \begin{cases} 
-4 - \theta/2, & \theta < -2 \\
3\theta/2, & -2 \leq \theta \leq 2 \\
4 - \theta/2, & \theta > 2 
\end{cases} \tag{3.15}
\]

as shown by the solid line in Figure 8.
Figure 8: The function $F(\theta, t)$ (3.15) for $t = 0$ (solid), $t = 1/8$ (dotted), and $t = 3/8$ (dashed).

The unique solution of $F(\theta, t) = 0$ in $(-4, 4)$ is $\theta_\ast(t) = -(2/3)\sin(4\pi t)$. A dynamic inverse of $F(\theta, t)$ is $G[w, \theta, t] = w$ corresponding to $\beta = 1$. A dynamic inverter for $F$ is then

$$\dot{\theta} = -\mu F(\theta, t)$$  \hfill (3.16)

for any real constant $\mu > 0$, where $F(\theta, t)$ is defined by (3.15). For this example we take $E(\theta, t) \equiv 0$, though in a later example, Example 3.7, we will construct and use a non-zero $E(\theta, t)$.
Figure 9: The upper graph shows the solutions of the dynamic inverter (3.16) for $\mu = 10$ (dashed) and $\mu = 100$ (solid). The initial condition was $\theta(0) = 3$. The lower graph shows the estimation error for the dynamic inverter (3.16) using $\mu = 10$ (dashed) and $\mu = 100$ (solid).

The top graph of Figure 9 shows the simulated solutions of (3.16) for $\theta(0) =$
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3, with $\mu = 10$ and $\mu = 100$. The simulations were done in Matlab [22] using an adaptive step-size fourth and fifth order Runge-Kutta integrator ode45. Each solution can be seen to converge to a neighborhood of $\theta_*(t)$; the higher the value of $\mu$, the smaller the neighborhood and the faster the convergence. The estimation error for each of the simulations is shown in the bottom graph of Figure 9.

As an analog computational paradigm, it is natural to consider the realization of a dynamic inverter in an analog circuit.

**Example 3.4 A Dynamic Inverter Circuit.** Consider a nonlinear circuit element, such as a diode, represented schematically in Figure 10.

![](image1.png)

Figure 10: Nonlinear circuit element of Example 3.4.

Assume that the circuit element is characterized by

$$i = f(V_a - V_b)$$

where $i$ is the current through the circuit element, and $V_a$ and $V_b$ are the voltages at each end of the circuit element as indicated in Figure 10. Assume that the characteristic of the circuit element is continuous, strictly monotonic, and lies in the shaded region of the graph of Figure 11.

![](image2.png)

Figure 11: The characteristic $V_a - V_b$ versus $f(V_a - V_b)$ is strictly monotonic, continuous, and lies in the shaded region. A typical curve is shown. See Example 3.4.

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Now consider the circuit of Figure 12 composed of linear resistors, ideal operational amplifiers, and the nonlinear circuit element with characteristic (3.17).

![Circuit Diagram](Image)

Figure 12: Circuit realization of a dynamic inverter. See Example 3.4.

The dynamic inverter is composed of a number of standard operational amplifier circuits: an integrator, a current to voltage converter, an inverting amplifier, and a differential amplifier. For a review of the characteristics of such circuits see Chua, Desoer, and Kuh [23], Chapter 4. To solve for \( V_{\text{out}} \) in terms of \( V_{\text{in}} \) note the following:

\[
V_{\text{out}} = -\frac{1}{R_0 C} \int_0^t V_3 \, dt + V_{\text{out}}(0)
\]

(3.18)

where \( V_{\text{out}}(0) \) is due to any charge on the capacitor at \( t = 0 \),

\[
V_3 = \frac{R_3}{R_2} (V_2 - V_1)
\]

(3.19)

\[
V_1 = -R_1 i
\]

(3.20)

\[
V_2 = \frac{R_5}{R_4} V_{\text{in}}
\]

(3.21)

\[
i = f(V_{\text{out}})
\]

(3.22)

---

7For an ideal operational amplifier, the open-loop gain of the amplifier is infinite, and no current flows into the + or − terminals. See [23], page 175.
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Substitute Equations (3.20), (3.21), and (3.22) into Equation (3.19), let

\[ R_1 = \frac{R_5}{R_4} \]  
(3.23)

and then substitute the resulting expression for \( V_3 \) into (3.18) to get

\[ V_{out} = -\frac{R_3 R_5}{R_2 R_4 R_6} \int_0^t (f(V_{out}) - V_{in}) \, dt + V_{out}(0) \]  
(3.24)

Let

\[ \mu = -\frac{R_3 R_5}{R_2 R_4 R_6} \]  
(3.25)

Now differentiate Equation (3.24) to get

\[ \dot{V}_{out} = -\mu (f(V_{out}) - V_{in}) \]  
(3.26)

The differential equation (3.26) is a dynamic inverter which solves for \( \theta \) satisfying \( f(\theta, t) = V_{in} \), thus the circuit of Figure 12 is a realization of a dynamic inverter for the nonlinear circuit elements characteristic \( F(V_4 - V_6) \). For sufficiently high \( \mu \), \( \| V_{in}(\cdot) \|_\infty \) sufficiently small, and after a transient, the relationship between \( V_{in} \) and \( V_{out} \) is approximately characterized by the inverse of the characteristic of the nonlinear circuit element as indicated in Figure 13. The larger the value of \( \mu \) and the smaller the bound \( \| \dot{V}_{in}(\cdot) \|_\infty \), the better the relation between \( V_{in} \) and \( V_{out} \) approximates the inverse of the nonlinear characteristic.

![Figure 13: Effective characteristic (solid) of the dynamic inverter circuit of Figure 12. The nonlinear element's characteristic is indicated in gray. See Example 3.4.](image)

Of course, practical realizations of such circuits as that of Figure 12 normally require modification in order to compensate for temperature fluctuations and non-ideal properties of the operational amplifiers. ▲
3.2 Dynamic Inversion with Vanishing Error

It is often the case that a differentiable representation of $F(\theta, t)$ is available. Under this condition an estimator, $E(\theta, t)$, for $\hat{\theta}_*(t)$ may be obtained. Differentiate $F(\theta_*(t), t) = 0$ with respect to $t$ to get the identity

$$D_1 F(\theta_*(t), t) \dot{\theta}_*(t) + D_2 F(\theta_*(t), t) = 0$$

(3.27)

Solve for $\dot{\theta}_*$, and replace $\dot{\theta}_*$ by $\theta$ to get the derivative estimator

$$E(\theta, t) := -D_1 F(\theta, t)^{-1} D_2 F(\theta, t)$$

(3.28)

If $F(\theta, t)$ is $C^2$ in $\theta$, then $E(\theta, t)$ becomes arbitrarily precise estimator of $\dot{\theta}_*(t)$ as $\theta$ approaches $\theta_*(t)$. Other approximators $E(\theta, t)$ satisfying $E(\theta, t) \to \theta_*$ as $\theta \to \theta_*$ are also possible as we will see in Section 4. Derivative estimators of this sort may be incorporated into dynamic inversion in order to produce an estimator $\theta(t)$ for $\theta_*(t)$ that is not only attracted to a neighborhood of $\theta_*(t)$ in finite time, as in the case of Theorem 3.1, but is attracted to $\theta_*(t)$ itself as $t \to \infty$. We will formalize this fact in the following theorem.

**Theorem 3.5 Dynamic Inversion Theorem – Vanishing Error.** Let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$, with $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(\theta, t) \mapsto F(\theta, t)$. Assume that $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(w, \theta, t) \mapsto G(w, \theta, t)$, is a dynamic inverse of $F(\theta, t)$ on $\mathcal{B}_r$, for some finite $\beta > 0$. Let $E : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(\theta, t) \mapsto E(\theta, t)$ be locally Lipschitz in $\theta$ and continuous in $t$. Assume that for some constant $\kappa \in (0, \infty)$, $E(\theta, t)$ satisfies

$$\left\| E(z + \theta_*(t), t) - \dot{\theta}_*(t) \right\|_2 \leq \kappa \|z\|_2$$

(3.29)

for all $z \in \mathcal{B}_r$. Let $\theta(t)$ denote the solution to the system

$$\dot{\theta} = -\mu G[F(\theta(t), t), \theta(t)] + E(\theta(t))$$

(3.30)

with initial condition $\theta(0)$ satisfying $\theta(0) - \theta_*(0) \in \mathcal{B}_r$. Then

$$\|\theta(t) - \theta_*(t)\|_2 \leq \|\theta(0) - \theta_*(0)\|_2 e^{-(\mu\beta - \kappa)t}$$

(3.31)

for all $t \in \mathbb{R}_+$, and in particular if $\mu > \kappa / \beta$, then $\theta(t)$ converges to $\theta_*(t)$ exponentially as $t \to \infty$. \hfill \blacktriangle
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Proof of Theorem 3.5: As before, let \( z(t) := \theta(t) - \theta_\ast(t) \), \( \tilde{F}(z, t) := F(z + \theta_\ast(t), t) \), and \( G[w, z, t] := G[w, z + \theta_\ast(t), t] \). Differentiate \( z(t) = \theta(t) - \theta_\ast(t) \) with respect to \( t \), and substitute (3.30) for \( \theta \) to get

\[
\dot{z} = -\mu \tilde{G} \left[ \tilde{F}(z, t), z, t \right] + E (z + \theta_\ast(t), t) - \dot{\theta}_\ast(t)
\]

(3.32)

Let

\[
V(z) := \frac{1}{2} \|z\|^2
\]

(3.33)

Differentiate \( V \) with respect to \( t \) to get

\[
\frac{d}{dt} V(z) = -\mu z^T \tilde{G} \left[ \tilde{F}(z, t), z, t \right] + z^T \left( E (z + \theta_\ast(t), t) - \dot{\theta}_\ast(t) \right)
\]

(3.34)

Then by Definition 2.1 and (3.29),

\[
\frac{d}{dt} V(z) \leq -\mu \beta \|z\|^2 + k \|z\|^2 = -(\mu \beta - k) \|z\|^2
\]

(3.35)

so that for \( z \in B_r \),

\[
\frac{d}{dt} V(z) \leq -2(\mu \beta - k)V(z)
\]

(3.36)

It follows that

\[
V(z) \leq V(0)e^{-2(\mu \beta - k)t}
\]

(3.37)

Substitute the right-hand side of (3.33) for \( V(z) \) to conclude

\[
\|z(t)\| \leq \|z(0)\|e^{-\mu \beta - k}t
\]

(3.38)

\[\blacksquare\]

Remark 3.6 Note that as in Theorem 3.1, differentiability of \( F(\theta, t) \) is not required for application of Theorem 3.5, though when \( F(\theta, t) \) is differentiable we may construct \( E(\theta, t) \) as in (3.28).

Example 3.7 Dynamic Inversion of Piecewise Linear Map – With \( \dot{\theta}_\ast \) Estimate. Let \( F \) and \( G \) be as in Example 3.3. Let \( E(\theta, t) := -(8\pi/3) \cos(4\pi t) \) which is the time-derivative of the solution \( \theta_\ast = -(4\pi/3) \sin(4\pi t) \) to \( F(\theta, t) = 0 \). Use the dynamic inverter

\[
\dot{\theta} = -\mu F(\theta, t) + E(\theta, t)
\]

(3.39)

with the same initial condition as before, \( \theta(0) = 3 \). The top graph of Figure 14 shows the simulation results, and the bottom graph of Figure 14 shows the estimation error. In this case the errors can be seen to go to zero exponentially.

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Figure 14: The top graph shows solutions of the dynamic inverter (3.39) with $E(\theta, t) = \theta(t)$ for $\mu = 10$ (dashed) and $\mu = 100$ (solid), with the actual solution $\theta(t)$ (dotted). The initial condition was $\theta(0) = 3$. The bottom graph shows the corresponding estimation error.

Note that for both dynamic inverters (3.16) and (3.39), each of the solutions $\theta(t)$ pass through the point $\theta = 2$, a local maximum of $F$, for which $F(\theta(t), t)$ is
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not differentiable. In contrast, Newton’s method and the gradient method are undefined for non-differentiable functions, and even if we were to make $F(\theta, t)$ differentiable in $\theta$ by smoothing it, Newton’s method would fail due to the local maximum at $\theta = 2$. ▲

**Remark 3.8 Dynamic Inversion with Perfect Initial Conditions.** If $\theta(0) = \theta_*(0)$, then the conditions of Theorem 3.5 guarantee that $\theta(t) \equiv \theta_*(t)$ for all $t \in \mathbb{R}_+$. So in a sense, we need only solve the inverse problem at a single instant $t = 0$. Then the dynamic inversion takes care of maintaining the inversion for all $t$. ▲

**Remark 3.9 Maintenance of a State-Dependent Jacobian Inverse as Dynamic Inverse.** Let

$$G[w, \theta, t] := D_1 F(\theta, t)^{-1} \cdot w \quad (3.40)$$

where $\theta$ is the state of a dynamic inverter. It follows from Lemma 2.11 and Theorem 3.5 that if $\mu$ is sufficiently large, $||\theta(0) - \theta_*(0)||$ is sufficiently small, and $G[w, \theta(0), 0]$ is a dynamic inverse of $F(\theta, t)$ at $t = 0$, then $G[w, \theta, t]$ is a dynamic inverse of $F(\theta, t)$ for all $t > 0$ (See also Remark 2.15). ▲

Example 3.10 will illustrate application of Remark 3.9 to the estimation of $\theta_*(t)$.

**Example 3.10 Dynamic Inversion Using State-Dependent Jacobian Inverse.** Let $w$ and $\theta$ be in $\mathbb{R}^n$. Assume that the assumptions of Lemma 2.11 hold. We may obtain an estimator $E(\theta, t)$ for $\theta_*$ from (3.28). Assume that $r$ has been chosen sufficiently small, and that $D_2 F(\theta, t)$ is sufficiently bounded so that $E(\theta, t)$ satisfies (3.29) for all $z \in B_r$. Let

$$G[w, \theta, t] := D_1 F(\theta, t)^{-1} \cdot w \quad (3.41)$$

and assume that $r$ is small enough that $G$ is a dynamic inverse of $F$ on $B_r$. If $(\theta(0) - \theta_*(0)) \in B_r$, and $\mu$ is sufficiently large, then by Theorem 3.5 the approximation error $z(t) := \theta(t) - \theta_*(t)$ using (3.30) will converge exponentially to zero. ▲

4 Dynamic Estimation of a Dynamic Inverse

In this section we will show how we can apply the dynamic inversion theorem to the construction of a dynamic system whose state includes both a dynamic inverse of a particular $F$ as well as an approximation for the root of $F$. Consideration of the example of dynamic inversion of a time-varying matrix [24] will lead the way.

---

8 In [24, 20] we cover dynamic matrix inversion in more depth.
Example 4.1 Inversion of Time-Varying Matrices. Consider the problem of estimating the inverse $\Gamma^*(t) \in \mathbb{R}^{n \times n}$ of a time-varying matrix $A(t) \in GL(n, \mathbb{R})$, where $GL(n, \mathbb{R})$ denotes the group of invertible matrices in $\mathbb{R}^{n \times n}$. Assume that we have representations for both $A(t)$ and $A(t)$, and that $A(t)$ is $C^1$ in $t$. Let $\Gamma$ be an element of $\mathbb{R}^{n \times n}$.

In order for $\Gamma^*$ to be the inverse of $A(t)$, $\Gamma^*$ must satisfy

$$A(t)\Gamma - I = 0 \quad (4.1)$$

Let $F : \mathbb{R}^{n \times n} \times \mathbb{R}^+ \to \mathbb{R}^{n \times n}; (\Gamma, t) \mapsto F(\Gamma, t)$ be defined by

$$F(\Gamma, t) := A(t)\Gamma - I \quad (4.2)$$

As usual we will refer to the solution of $F(\Gamma, t) = 0$ as $\Gamma^*(t)$. To obtain an estimator $E(\Gamma, t)$ for $\Gamma^*(t)$, differentiate $A\Gamma = I$ with respect to $t$, solve the resulting expression for $\Gamma^*$, replace $A^{-1}$ by $\theta^*$, and then replace $\Gamma^*$ by $\Gamma$ in the resulting expression to get

$$E(\Gamma, t) := -\dot{\Gamma}A(t)\Gamma \quad (4.3)$$

Differentiate $F(\Gamma, t)$ with respect to $\Gamma$ to get

$$D_\Gamma F(\Gamma, t) = A(t) \quad (4.4)$$

whose inverse is $\Gamma^*$. So a choice of dynamic inverse is

$$G[w, \Gamma, \dot{\Gamma}] := \Gamma \cdot w \quad (4.5)$$

for $\Gamma$ sufficiently close to $\Gamma^* = A^{-1}(t)$ and with $w \in \mathbb{R}^{n \times n}$. The dynamic inverter for this problem then takes the form

$$\dot{\Gamma} = -\mu G[F(\Gamma, t), \Gamma] + E(\Gamma, t) \quad (4.6)$$

or, expanded,

$$\dot{\Gamma} = -\mu \Gamma (A(t)\Gamma - I) - \Gamma \dot{A}(t)\Gamma \quad (4.7)$$
and we choose as initial conditions $I(0) \approx I_\star(0) = A^{-1}(0)$ so that the estimation error starts small. Theorem 3.5 guarantees that for sufficiently large $\mu$, and for $I(0)$ sufficiently close to $A^{-1}(0)$, equation (4.7) will produce an estimator $I(t)$ whose error $\|I(t) - I_\star(t)\|$ decays exponentially to zero at a rate determined by our choice of $\mu$. Even if we don’t know $A(t)$, we can, by Theorem 3.1, take $E(I; t)$ to be identically zero and achieve inversion with a bounded error.

\[ \blacksquare \]

**Remark 4.2 Inversion of Time-Varying Matrices.** Example 4.1 allows one to invert time-varying matrices without calling upon discrete matrix inversion routines. One only need calculate or approximate a single inverse, $A(0)^{-1}$. The flow of (4.7) then takes care of the inversion for all $t > 0$.

\[ \blacksquare \]

**Remark 4.3 Notation.** In the following example and in the remainder of this section we will couple two dynamic inverters together; one which estimates the solution $\theta_\star$ of $F(\theta, t) = 0$, and the other which solves for a matrix $I$ to be used in a dynamic inverse $G[w, I]$ of $F(\theta, t)$. In order to distinguish between the map, dynamic inverse, and derivative estimator for each of the two problems we adopt the convention of referring to the map, dynamic inverse, and derivative estimator for $I$ by $F^\gamma$, $G^\gamma$, and $E^\gamma$ respectively, retaining the designations $F$, $G$, and $E$ for the map, dynamic inverse, and derivative estimator for $\theta$.

\[ \blacksquare \]

**Example 4.4 Obtaining a Dynamic Inverse Dynamically.** Assume that $F(\theta, t)$ satisfies the assumptions of Lemma 2.11 with continuous isolated solution $\theta_\star$. Assume that $D_1F(\theta, t)$ is $C^2$ in $\theta$ and $C^1$ in $t$. Let $I \in \mathbb{R}^{n \times n}$ denote an estimator for $D_1F(\theta_\star(t))^{-1}$. We may then estimate $\theta_\star(t)$ as follows: Differentiate $F(\theta_\star, t) = 0$, solve for $\theta_\star$, and substitute $I$ for $D_1F(\theta_\star(t), t)^{-1}$ and $\theta$ for $\theta_\star$ to obtain an estimator for $\theta_\star$ in terms of $\gamma, \theta$, and $t$.

\[
E(I, \theta, t) := -\Gamma D_2F(\theta, t) \tag{4.8}
\]

Assume that $E(I, \theta, t)$ is Lipschitz in $I$ and $\theta$, and piecewise-continuous in $t$. Using $E(I, \theta, t) = [E_i(I, \theta, t)]_{i \in \mathbb{N}}$, and in a manner similar to (4.3) we estimate $\Gamma_\star$ with

\[
E^\gamma(I, \theta, t) := -\Gamma \frac{d}{dt} D_1F(\theta, t) \bigg|_{\theta = E(I, \theta, t)} \Gamma \tag{4.9}
\]
(see (4.3)) where
\[ \frac{d}{dt} \frac{D_1 F(\theta, t)}{\theta = \mathcal{P} \Gamma(t, \theta, t)} := \sum_{i=1}^{n} \frac{\partial D_1 F(\theta, t)}{\partial \theta_i} E_i(\Gamma, \theta, t) + \frac{\partial D_1 F(\theta, t)}{\partial t} \]
In this case,
\[ F'(\Gamma, \theta, t) := D_1 F(\theta, t) \Gamma - I \]  
(4.11)
Let
\[ G^\Gamma[W; \Gamma] := \Gamma \cdot W \]
(4.12)
as in Example 4.1 (see (4.5)), with \( W \in \mathbb{R}^{n \times n} \).

Theorem 3.5 now tells us that we may estimate \( \theta_*(t) \) with the system of coupled nonlinear differential equations
\[ \left[ \begin{array}{c} \dot{\Gamma} \\ \dot{\theta} \end{array} \right] = -\mu \left[ \begin{array}{cc} \Gamma & 0 \\ 0 & \Gamma \end{array} \right] \cdot \left[ \begin{array}{c} F'(\Gamma, \theta, t) \\ F(\theta, t) \end{array} \right] + \left[ \begin{array}{c} E'(\Gamma, \theta, t) \\ E(\Gamma, \theta, t) \end{array} \right] \]
(4.13)
with guaranteed exponential convergence of \( (\Gamma, \theta) \) to \( (\Gamma_*, \theta_*) \).

After a definition, we summarize the result of Example 4.4 with a theorem.

**Definition 4.5** For \( (\Gamma, \theta) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n \), define the norm \( \|(\Gamma, \theta)\|_2 \) by
\[ \|(\Gamma, \theta)\|_2 := \left( \sum_{i=1}^{n} |\Gamma_{ij}|^2 + \sum_{i=1}^{n} |\theta_i|^2 \right)^{1/2} \]
(4.14)
Norm (4.14) is thus the \( l_2 \) norm of the matrix \([\Gamma, \theta]\) where we consider \( \theta \) to be a column vector.

**Theorem 4.6 Dynamic Inversion with Dynamic Determination of a Dynamic Inverse.** Let \( F(\theta, t) \) satisfy the assumptions of Lemma 2.11. Then for \( \Gamma(0) \) sufficiently close to \( D_1 F(\theta_*, 0)^{-1} \), and \( \theta(0) \) sufficiently close to \( \theta_*(0) \), the solution \((\Gamma(t), \theta(t))\) of
\[ \left[ \begin{array}{c} \dot{\Gamma} \\ \dot{\theta} \end{array} \right] = -\mu \left[ \begin{array}{cc} \Gamma & 0 \\ 0 & \Gamma \end{array} \right] \left[ \begin{array}{c} D_1 F(\theta, t) \Gamma - I \\ F(\theta, t) \end{array} \right] + \left[ \begin{array}{c} -\Gamma' D_1 F(\theta, t) \theta \{ F(\theta, t) \} \\ -\Gamma D_2 F(\theta, t) \end{array} \right] \]
(4.15)
satisfies $(\Gamma(t), \theta(t)) \to (D_1 F(\theta_*, t)^{-1}, \theta_*(t))$ as $t \to \infty$. Furthermore, for sufficiently large $\mu > 0$, the convergence is exponential, i.e., there exist $k_1 > 0$ and $k_2 > 0$ such that

$$
\|((\Gamma(t), \theta(t)) - (\Gamma_*(t), \theta_*(t))\|_2 \leq k_1 \|((\Gamma(0), \theta(0)) - (\Gamma_*(0), \theta_*(0))\|_2 e^{-k_2 t}
$$

(4.16)

for all $t \geq 0$, where $\Gamma_*(t) = D_1 F(\theta_*(t), t)^{-1}$.

\textbf{Proof of Theorem 4.6:} Let

$$
\hat{G}([W, w], \Gamma) := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} W \\ w \end{bmatrix}
$$

(4.17)

and

$$
\hat{F}(\Gamma, \theta, t) := \begin{bmatrix} F^T(\Gamma, \theta, t) \\ F(\theta, t) \end{bmatrix}
$$

(4.18)

Then $\hat{G}([W, w], \Gamma_*) = D_1 \hat{F}(\Gamma_*, \theta_*, t)^{-1} \cdot [W^T, w^T]^T$, and $\hat{G}([W, w], \Gamma_*)$ is continuous in its arguments. Hence, for $\Gamma(0)$ sufficiently close to $D_1 F(\theta_*, 0)^{-1}$, and $\theta(0)$ sufficiently close to $\theta_*(0)$, $\hat{G}([W, w], \Gamma)$ is a dynamic inverse of $\hat{F}(\Gamma, \theta, t)$.

Also for

$$
\hat{E}(\Gamma, \theta, t) := \begin{bmatrix} E^T(\Gamma, \theta, t) \\ E(\Gamma, \theta, t) \end{bmatrix}
$$

(4.19)

we have that $\hat{E}(\Gamma_*, \theta_*) = (\hat{I}_*, \hat{\theta}_*)$ and $\hat{E}(\Gamma_*, \theta_*, t)$ is continuous in $(\Gamma, \theta)$. Therefore, by Theorem 3.5, for sufficiently large $\mu > 0$, equation (4.15) is a dynamic inverter for $(\Gamma, \theta)$, and $(\Gamma, \theta)$ converges exponentially to $(\Gamma_*, \theta_*)$.

\textbf{Example 4.7 Tracking an Intersection of Two Time-Varying Curves.} Consider the two time-dependent cubic curves in the $x, y$ plane,

$$
\begin{align*}
y &= (2 + \sin(t))x^3 + (-1 + \frac{1}{2} \sin(\sqrt{2}t))x \\
y &= -(2 + \sin(3t))x^3 + (1 + \frac{1}{4} \sin^2(5t))x
\end{align*}
$$

(4.20)

For each $t \geq 0$ it is readily verified that these curves intersect at three points: one point is the origin, one is to the right of $(0, 0)$ at $\theta_*(t) = (x_*(t), y_*(t))$, and one is to the left of the origin at $-(x_*(t), y_*(t))$. Figure 15 shows the two curves and their intersections for six values of $t$ and for $x \geq 0$. 
Figure 15: The solution of interest in Example 4.7, \( \theta_* (t) = (x_*(t), y_*(t)) \), is the intersection (to the right of \((0,0)\)) of the two cubic curves shown in each of the graphs. This figure shows the pair of cubic curves (4.20) for \( t \in \{0, 1, \ldots, 5\} \).

Let
\[
F(\theta, t) = F(x, y, t) = \begin{bmatrix}
y - (2 + \sin(t))x^3 - \left(-1 + \frac{1}{3}\sin(\sqrt{2}t)\right)x \\
y + (2 + \sin(3t))x^3 - \left(1 + \frac{1}{3}\sin^2(5t)\right)x
\end{bmatrix}
\]  
(4.21)
We will be interested in the solution \( \theta_* (t) = (x_*(t), y_*(t)) \) of \( F(\theta, t) = 0 \) to the right of \( x = 0 \); the other solutions are \( (x_*(t), y_*(t)) = (0, 0) \) and \( (x_*(t), y_*(t)) = -\theta_* (t) \). We will use Theorem 4.6 to track the solution \( \theta_* (t) \).

In this case, \( \Gamma \in \mathbb{R}^{2 \times 2} \),
\[
D_1 F((x, y), t) = \begin{bmatrix}
-3(2 + \sin(t))x^2 - \left(-1 + \frac{1}{3}\sin(\sqrt{2}t)\right) & 1 \\
3(2 + \sin(3t))x^2 - \left(1 + \frac{1}{3}\sin^2(5t)\right) & 1
\end{bmatrix}
\]  
(4.22)
\[
D_2 F((x, y), t) = \begin{bmatrix}
-\cos(t)x^3 - \frac{\sqrt{2}}{3}\cos(\sqrt{2}t)x \\
3\cos(3t)x^3 - \frac{5}{3}\sin(5t)\cos(5t)x
\end{bmatrix}
\]  
(4.23)
\[
E(\Gamma, (x, y), t) = -\Gamma D_2 F((x, y), t) = \begin{bmatrix}
E^1 \\
E^2
\end{bmatrix}
\]  
(4.24)

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and

\[
\frac{d}{dt} D_t F((x, y), t) = \begin{bmatrix}
-3 \cos(t)x^2 - 6(2 + \sin(t))x E^1 - \sqrt{2} \cos(\sqrt{2} t) & 0 \\
9 \cos(3t)x^2 + 6(2 + \sin(3t))x E^1 - \frac{9}{2} \sin(5t) \cos(5t) & 0
\end{bmatrix} \tag{4.25}
\]

From (4.21), (4.22), (4.23), and (4.25) we can construct the dynamic inverter (4.15).

When \( t = 0 \), the root (to the right of \((0, 0)\)) can be obtained by inspection as \((x_*(0), y_*(0)) = (\frac{1}{\sqrt{2}}, 0)\). Thus we could use \((x(0), y(0)) = (1/\sqrt{2}, 0)\) and \( \Gamma(0) = D_t F((x(0), y(0)), t)^{-1} \) for initial conditions for the dynamic inverter to produce the exact\(^9\) solution \((x_*(t), y_*(t))\) for all \( t \geq 0 \). In order to demonstrate an error transient, however, we choose initial conditions \((x(0), y(0)) = (1, 0)\) and

\[
\begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix}, \quad \Gamma(0) = \begin{bmatrix}
-1/4 & 1/4 \\
1/2 & 1/2
\end{bmatrix} \tag{4.26}
\]

Figure 16 shows the results of a simulation of the dynamic inverter using the adaptive step-size Runge-Kutta integrator \texttt{ode45} from Matlab [22], with \( \mu = 10 \). The upper graph shows \( x(t) \) (solid) and \( y(t) \) (dashed) versus \( t \). The lower graph shows \((x(t), y(t))\) for \( t \in [0, 10] \). The root \( \theta_*(t) = (x_*(t), y_*(t)) \) is a quasi-periodic curve. Note that if we were to change \( \sqrt{2} \) to \( 2 \), for instance, in (4.20) and (4.21), the solution would have a period of \( 2\pi \).

\(^9\)By “exact” for the simulation, we mean exact up to the tolerance of the integrator which, in this example, was \( 10^{-6} \).
Figure 16: The solution of the dynamic inverter of Example 4.7 for $F(\theta, t) = 0$ corresponding to Example 4.7, where $\theta = (x, y)$. The upper graph shows $x(t)$ versus $t$ (solid) and $y(t)$ versus $t$ (dashed). The lower graph shows $x(t)$ versus $y(t)$ with the initial condition $(x(0), y(0)) = (1,0)$ marked by the small circle.

Figure 17 shows the log of the approximation error as seen through $F$, namely $\log_{10} \| F((x(t), y(t)), t) \|_{\infty}$. The error can be seen to decay to the level of the
integrator tolerance, $10^{-6}$, within 2 seconds.

Figure 17: The estimation error for the dynamic inverter of Example 4.7 as seen through $F$ (4.21), $\log_{10} \|F(\theta(t), t)\|_{\infty}$ versus $t$ in seconds. See Example 4.7.

A simple example of application of Theorem 4.6 to the inversion of robot kinematics appears in [20].

In the closing example of this chapter we apply Theorem 4.6 to the solution of a standard problem in the control of nonlinear systems.

**Example 4.8 Dynamic Inversion of a Nonlinear Control System.** Consider the multi-input, multi-output, time-varying nonlinear control system

$$\dot{x} = f(x, t, u)$$  \hspace{1cm} (4.27)

with $x$ and $u$ in $\mathbb{R}^n$. Assume that $f(x, t, u)$ is $C^2$ in its arguments. Assume also that $D_3 f(x, t, u)$ is invertible for all $(x, u)$ in a neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^n$ and for all $t \geq 0$. Consider also the vector field $\phi(x, t)$ assumed to be $C^2$ in $x$ and $t$. Suppose we wish to solve for $u$ such that

$$\phi(x, t) = f(x, t, u)$$  \hspace{1cm} (4.28)

i.e. we wish to solve for a $u(\cdot)$ that will cause the state $x(t)$ to obey the dynamics

$$\dot{x} = \phi(x, t)$$  \hspace{1cm} (4.29)
Define
\[ F(u, x, t) := f(x, t, u) - \phi(x, t) \]  \hspace{1cm} (4.30)

Let \( u_*(t) \) be a continuous isolated solution, assumed to exist, of \( F(u, x, t) = 0 \).

Let
\[ F^\gamma(\Gamma, u, x) := D_3 f(x, t, u) \Gamma - I \]  \hspace{1cm} (4.31)

so that \( \Gamma_*(t) = D_3 f(x, t, u_*)^{-1} \). As in Theorem 4.6 let
\[ G[w] = \Gamma \cdot w \]  \hspace{1cm} (4.32)

with \( w \in \mathbb{R}^n \), and
\[ G^\gamma[W] = \Gamma \cdot W \]  \hspace{1cm} (4.33)

with \( W \in \mathbb{R}^{n \times n} \). To solve for an estimator \( E(\Gamma, u, x, t) \) for \( u_* \), we differentiate \( F(u, x, t) = 0 \) with respect to \( t \), solve for \( \dot{u} \), and replace \( D_1 F(u, x, t)^{-1} \) by \( \Gamma \) and \( \dot{x} \) by \( f(x, u) \) to get
\[ E(\Gamma, u, x, t) = -\Gamma \left( (D_1 f(x, t, u) - D_1 \phi(x, t)) f(x, t, u) \right. \\
+ \left. D_2 f(x, t, u) - D_2 \phi(x, t) \right) \]  \hspace{1cm} (4.34)

Similarly, to solve for an estimator \( E^\gamma(\Gamma, u, x, t) \) for \( \dot{u}_* \), differentiate \( F^\gamma(\Gamma, u, x, t) = 0 \) with respect to \( t \), solve for \( \Gamma \), replace \( D_2 f(x, t, u)^{-1} \) by \( \dot{\gamma} \) by \( f(x, t, u) \), and \( \dot{u} \) by \( E(\Gamma, u, x, t) \) to get
\[ E^\gamma(\Gamma, u, x, t) = -\Gamma \left( (D_1, \dot{\gamma} f(x, t, u) - f(x, t, u) + D_2, \dot{\gamma} f(x, t, u) \right) \\
+ \left. D_3, \dot{\gamma} f(x, t, u) \right) \cdot E(\Gamma, u, x, t) \]  \hspace{1cm} (4.35)

Then if \( u(0) \) and \( \Gamma(0) \) are sufficiently close to \( u_*(0) \) and \( D_3 f(x(0), 0, u_*(0))^{-1} \) respectively, then a dynamic compensator which produces a \( u \) that converges exponentially toward \( u_* \) is
\[ \begin{cases} 
\dot{\Gamma} = -\mu \Gamma \cdot F^\gamma(\Gamma, u, x, t) + E^\gamma(\Gamma, u, x, t) \\
\dot{u} = -\mu \Gamma \cdot F(u, x, t) + E(\Gamma, u, x, t) 
\end{cases} \]  \hspace{1cm} (4.36)

Furthermore, if we choose \( u(0) \) to satisfy \( F(u(0), 0, 0) = 0 \), and \( \Gamma(0) \) to satisfy \( \Gamma(0) = D_3 f(x(0), 0, u(0))^{-1} \), then \( x(t) \) will satisfy (4.29) for all \( t \geq 0 \). Figure 18 shows the closed-loop system including the dynamic inversion compensator, and the original nonlinear plant (4.27).

![Dynamic Inversion Diagram](image-url)

Figure 18: The closed-loop system with dynamic inversion compensator (4.36) with state \( (\Gamma, u) \) and the nonlinear plant (4.27) with state \( x \).
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Note that if a convenient closed form exists for $(D_3 f(x, t, u))^{-1}$, or if one is satisfied to use discrete numerical matrix inversion, one could replace $\Gamma$ by $(D_3 f(x, t, u))^{-1}$ and eliminate the $\Gamma$ equations.

5 Generalizations of Dynamic Inversion

The dynamic inversion theorems, Theorems 3.1 and 3.5, rely upon the use of a quadratic Lyapunov function, and indeed, the definition of a dynamic inverse, Definition 2.1, is tailored for association with a quadratic Lyapunov function. We may generalize dynamic inversion based on more general Lyapunov functions. For instance, consider the following definition.

Definition 5.1 General Dynamic Inverse. For $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(\theta, t) \mapsto F(\theta, t)$ let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$. A map $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(w, t) \mapsto G[w, \theta, t]$ is called a dynamic inverse of $F$ on the ball $B_r := \{ z \in \mathbb{R}^n | ||z|| \leq r \}$, $r > 0$, if

1. the map $G[F(\theta, t), \theta, t]$ is Lipschitz in $\theta$, piecewise-continuous in $t$, and
2. there exists a continuously differentiable function $V : [0, \infty) \times B_r \to \mathbb{R}$; $(t, z) \mapsto V(t, z)$ such that for all $z \in B_r$,

$$\alpha_1(||z||) \leq V(t, z) \leq \alpha_2(||z||)$$

$$D_1 V(t, z) + D_2 V(t, z) \left( \theta_*(t) - G[F(z + \theta_*, t), z + \theta_*, t], z + \theta_*, t \right) \leq -\alpha_3(||z||)$$

(5.1)

where $\alpha_1(\cdot), \alpha_2(\cdot)$, and $\alpha_3(\cdot)$ are of class $K$ on $[0, r)$.

A more general dynamic inversion theorem follows from Definition 5.1.

Theorem 5.2 General Dynamic Inversion Theorem. Let $\theta_*(t)$ be a continuous isolated solution of $F(\theta, t) = 0$, with $F : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(\theta, t) \mapsto F(\theta, t)$. Assume that $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$; $(w, \theta, t) \mapsto G[w, \theta, t]$, is a dynamic inverse (Definition 5.1) of $F(\theta, t)$ on $B_r$. Let $\theta(t)$ denote the solution to the system

$$\dot{\theta} = -G[F(\theta, t), \theta, t]$$

(5.3)

with initial condition $\theta(0)$ satisfying $\theta(0) = \theta_*(0) \in B_r$. If $\theta(0) = \theta_*(0)$ is in $B_r$, then $\theta \to \theta_*$ asymptotically.

Proof of Theorem 5.2: Since $G[w, \theta, t]$ is assumed to be a dynamic inverse of $F(\theta, t)$, there exists a function $V(t, z)$ satisfying (5.1) and (5.2). It follows (see [25], Theorem 4.1, page 169) that the origin $z = 0$ of the system

$$\dot{z} = -G[F(z + \theta_*(t), t), z + \theta_*(t), t] + \dot{\theta}_*(t)$$

(5.4)
is uniformly asymptotically stable. Thus $\theta(t) \rightarrow \theta_*(t)$ asymptotically as $t \rightarrow \infty$. 

Though it is readily apparent that Definition 5.1 leads to a more general dynamic inversion theorem with a simple proof, it also imposes the generally difficult requirement of finding a Lyapunov function in order to prove that $G$ is indeed a dynamic inverse of $F$. In contrast the dynamic inverse criterion of Definition 2.1 is often easily verified from familiarity with the inverse problem one is trying to solve. For instance, one often knows that $D_t F(\theta, t)$ is invertible for all $\theta$ sufficiently close to $\theta_*(t)$. In such cases Definition 2.1 leads easily to the constructive methods of, for instance, Theorem 4.6. What we would gain in generality by relying upon Definition 5.1 we would lose in ease of construction of dynamic inverters for a broad and useful set of inverse problems.

Another consideration in our choice of dynamic inverse definition is that it leads to exponentially stable systems. Exponentially stable systems are known to maintain their exponential stability under a wide variety of perturbations. This fact has been of profound value in the history of control theory, accounting, for instance, for the wide successes of the application of linear controllers to the control of nonlinear systems. When dynamic inverters are incorporated into control laws, this exponential stability allows one to call upon a variety of well-known results of stability theory in order to conclude exponential stability of the closed-loop control system. By retaining exponential stability of a closed-loop control system we allow that control system to retain a useful level of robustness with respect to perturbations and modeling errors.

6 Summary

A dynamic system for the tracking of roots of nonlinear time-dependent maps has been presented and shown to guarantee arbitrary accuracy of estimation within an arbitrarily short time interval. The notion of a dynamic inverse of a map has been introduced. We have shown a number of ways in which dynamic inverses may be obtained, perhaps the most powerful being through dynamic inversion itself, where the dynamic inverse is solved for at the same time it is being used to track the root of interest; in this method the process of solving for the inverse of the linearized map is built into the dynamical process itself. By building upon simple Lyapunov stability arguments we have constructed a class of nonlinear dynamic systems that can solve time-dependent finite-dimensional inverse problems. We have shown how derivative estimation can be used to make the difference between an ultimately bounded approximation error, and an approximation error that converges exponentially to zero.

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using joint-space errors for feedback. In [20] dynamic inversion is incorporated into tracking controllers for nonminimum phase systems. Applications include tracking control while retaining balance for both an inverted pendulum on a cart [27, 20], as well as for a bicycle [28, 20].

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