INTERNAL EQUILIBRIUM CONTROL OF A BICYCLE

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ABSTRACT

Internal equilibrium control is applied to the problem of path-tracking with balance for the bicycle using steering and rear-wheel torque as inputs. From the internal dynamics of the bicycle an internal equilibrium manifold, a submanifold of the state-space, is constructed. The internal equilibrium controller makes a neighborhood of the manifold attractive and invariant. This results in approximate tracking of time-parameterized paths in the plane while retaining balance.

1. The Model

The simple bicycle model to be controlled is illustrated in Figure 1. The bicycle location is parameterized by (x, y), the position of the rear-wheel contact with the ground. The **contact-line** is the directed line from the rear-wheel contact to the front-wheel contact. The **yaw-angle** θ of the bicycle is the angle the contact line makes with a parallel to the x-axis. The **roll-angle** α of the bicycle is the angle that the bicycle frame is rolled away from the vertical (the roll-angle shown in Figure 1 is in the negative direction). The inputs to the bicycle are the **steering angle** ϕ and the component, parallel to the contact line, of the reaction force τ^r of the ground on the rear wheel.

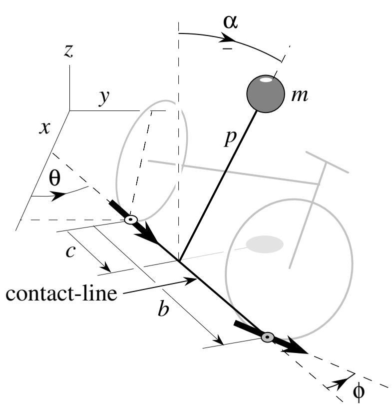

Figure 1: Bicycle model.

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Reduced Equations of Motion

Let $\sigma := \tan(\phi)/b$. Define v_r and $v_\perp$ (see Figure 2) by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} v_r \\ v_\perp \end{bmatrix} \quad (1)$$

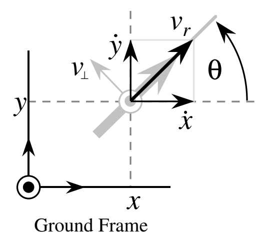

Figure 2: Relating v_r, $v_\perp$, $\dot{x}$, and $\dot{y}$.

The front and rear wheels of the bicycle are constrained to roll without slipping. We may express this constraint as

$$v_\perp = 0, \quad \dot{\theta} = v_r \sigma \quad (2)$$

It is convenient to define $w_\sigma := \dot{\sigma}$. Then, through various routes (see [1], [2]) the *reduced* equations of motion for the bicycle may be derived. They are

$$\begin{aligned} \dot{\sigma} &= w^\sigma \\ M(r) \begin{bmatrix} \ddot{\alpha} \\ \dot{v}_r \end{bmatrix} &= F(r) + B(r) \begin{bmatrix} w^\sigma \\ \tau^r \end{bmatrix} \end{aligned} \quad (3)$$

where

$$M(\alpha, \sigma) = \begin{bmatrix} p^2 & -cpc_\alpha \sigma \\ -cpc_\alpha \sigma & 1 + (c^2 + p^2 s_\alpha^2)\sigma^2 + 2p\sigma s_\alpha \end{bmatrix}$$

$$F(\alpha, \dot{\alpha}, \sigma, v_r) = \begin{bmatrix} gps_\alpha + (1 + p\sigma s_\alpha)pc_\alpha \sigma v_r^2 \\ -(1 + p\sigma s_\alpha)2pc_\alpha \sigma v_r \dot{\alpha} - cp\sigma s_\alpha \dot{\alpha}^2 \end{bmatrix}$$

$$B(\alpha, \sigma, v_r) = \begin{bmatrix} cpc_\alpha v_r & 0 \\ -(c^2\sigma + ps_\alpha(1 + p\sigma s_\alpha))v_r & 1/m \end{bmatrix}$$

The reduced equations (3) together with the constraints (2) and values of the inputs τ^r and w^σ provide a complete description of the motion of the bicycle.

2. Internal Equilibrium Control

Internal equilibrium control [3] is a method of tracking control for nonminimum phase systems. It will be illustrated here through application to the bicycle. For application to the inverted pendulum on a cart see [3, 4].

Through differentiation of the constraints, input prolongations, and a sequence of state-dependent coordinate changes (see [3] for details) equations (3) may be brought to the form

$$\left\{ \begin{array}{rcl} \begin{bmatrix} x^{(3)} \\ y^{(3)} \end{bmatrix} &=& \begin{bmatrix} -2\dot{v}_r s_\theta - v_r c_\theta \dot{\theta} \\ 2\dot{v}_r c_\theta - v_r s_\theta \dot{\theta} \end{bmatrix} \dot{\theta} \\ && + \begin{bmatrix} c_\theta & -v_r s_\theta \\ s_\theta & v_r c_\theta \end{bmatrix} \begin{bmatrix} u^r \\ u^\theta \end{bmatrix} \\ \ddot{\alpha} &=& \frac{g}{p} s_\alpha + \frac{1}{p}\left(1 + \frac{p\dot{\theta} s_\alpha}{v_r}\right) c_\alpha \dot{\theta} v_r + \frac{c}{p} c_\alpha u^\theta \end{array} \right. \tag{4}$$

where $u^r := \ddot{v}_r$, and $u^\theta := \ddot{\theta}$ are new inputs. Note that through the relations (1, 2) all variables in (4) may be expressed in terms of x, y and their derivatives with respect to t. The **internal dynamics** of (4) are the dynamics of the $\ddot{\alpha}$ equation in (4).

Let

$$\begin{array}{rcl} V_x &:=& x_d^{(3)} - \sum_{i=0}^{2} \gamma_i (x^{(i)} - x_{di}^{(i)}) \\ V_y &:=& y_d^{(3)} - \sum_{i=0}^{2} \gamma_i (y^{(i)} - y_{di}^{(i)}) \end{array} \tag{5}$$

where the γ_i's are chosen so that the roots of $s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0$ have negative real parts. In the absence of α-dynamics, we could make (x, y) exponentially converge to track any C^3 trajectory $(x_d(t), y_d(t))$, as long as $v_r \geq v_{r\min} > 0$, by choosing inputs u^r_{ext} and u^θ_{ext} to be

$$\begin{bmatrix} u^r_{\text{ext}} \\ u^\theta_{\text{ext}} \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta / v_r & c_\theta / v_r \end{bmatrix} \cdot \left(- \begin{bmatrix} -2\dot{v}_r s_\theta - v_r c_\theta \dot{\theta} \\ 2\dot{v}_r c_\theta - v_r s_\theta \dot{\theta} \end{bmatrix} \dot{\theta} + \begin{bmatrix} V_x \\ V_y \end{bmatrix} \right) \tag{6}$$

which, though substitution into (4), gives $x^{(3)} = V_x$, and $y^{(3)} = V_y$. The controller (6) is termed the **external tracking controller** for the bicycle.

Assume that $v_r = v_{rd}(t) > v_{r\min} > 0$ for some C^2 $v_{rd}(t)$. Let $\lambda_d(t) \in (-\pi/2, \pi/2)$ be C^2 for all $t \geq 0$. The control

$$\begin{array}{rl} u^\theta_{\text{int}}(v^\theta_{\text{int}}) & := \frac{p}{c\cos(\alpha)} \left(-\frac{g}{p}\sin(\alpha) \right. \\ & \left. -\frac{1}{p}\left(1 + \frac{p\dot{\theta}\sin(\alpha)}{v_r}\right)\cos(\alpha)\dot{\theta} v_r + v^\theta_{\text{int}} \right) \\ v^\theta_{\text{int}} & := \ddot{\lambda}_d - \beta_1(\dot{\alpha} - \dot{\lambda}_d) - \beta_0(\alpha - \lambda_d) \end{array} \tag{7}$$

where the roots of $s^2 + \gamma_1 s + \gamma_0$ have negative real parts, will cause α to converge to λ_d exponentially. Examples of resulting paths in the plane can be seen in [1]. Equation (7) is called an **internal tracking controller** for the bicycle.

The **internal equilibrium angle** is defined as the solution α_e to the equation obtained by setting $\dot{\alpha} = 0$, $\ddot{\alpha} = 0$, and $u^\theta = u^\theta_{\text{ext}}$ in the equation for the internal dynamics,

$$0 = g\tan(\alpha_e) + \left(1 + \frac{p\dot{\theta}\sin(\alpha_e)}{v_r}\right)\dot{\theta} v_r + c u^\theta_{\text{ext}} \tag{8}$$

For $z(t) \in \mathbb{R}$, let $z^{(0,k)}$ denote $(z^{(0)}, z^{(1)}, \ldots, z^{(k)})$ where $z^{(i)}$ is the i^{th} derivative of $z(t)$ with respect to t. Note that α_e is an implicit function of $x^{(0,2)}$, $y^{(0,2)}$, $x_d^{(0,3)}(t)$, and $y_d^{(0,3)}(t)$. Let $q_p = (x^{(0,2)}(t_p), y^{(0,2)}(t_p))$ at a particular time $t_p \geq 0$. This determines a value of $u^\theta_{\text{ext}}(q_p, t_p)$. The internal equilibrium angle is that roll angle α_e such that if u^θ were held *constant* at $u^\theta_{\text{ext}}(q_p, t_p)$, then $(\dot{\alpha}, \ddot{\alpha}) = 0$. In general α_e changes as the time and state changes. The **internal equilibrium manifold** is the state-space submanifold

$$\mathcal{E}(t) = \left\{ (x^{(0,2)}, y^{(0,2)}, \alpha^{(0,1)}) \mid \alpha = \alpha_e, \dot{\alpha} = 0 \right\} \tag{9}$$

Let F denote the vector field of (4) where $u^r = u^r_{\text{ext}}$ and $u^\theta = u^\theta_{\text{ext}}$. If (x, y) is approximately tracking (x_d, y_d) then the vector field associated with the (x, y)-dynamics of the bicycle are approximately F. Vector field F is the **nominal external dynamics**.

An **internal equilibrium controller** for the bicycle is obtained by letting $u^r = u^r_{\text{ext}}$, and

$$\begin{array}{rcl} u^\theta &=& u^\theta_{\text{ext}}(v_e) \\ v_e &:=& L_F^2 \alpha_e - \beta_1(\dot{\alpha} - L_F \alpha_e) - \beta_0(\alpha - \alpha_e) \end{array} \tag{10}$$

where $L_F^i \alpha_e$ is the i^{th} Lie derivative of α_e in the direction of F. An explicit estimator $\hat{\alpha}_e$ for the implicitly defined α_e is produced by the **dynamic inverter** [3]

$$\begin{array}{rcl} \dot{\hat{\alpha}}_e &=& -\mu\left(g\tan(\alpha_e) + \left(1 + \frac{p\dot{\theta}\sin(\alpha_e)}{v_r}\right)\dot{\theta} v_r + c u^\theta_{\text{ext}} \right) \\ && + L_F \alpha_e \end{array} \tag{11}$$

with $\mu > 0$. The internal equilibrium controller causes a neighborhood of $\mathcal{E}(t)$ to become attractive and invariant and thereby produces approximate tracking $(x, y) \to (x_d, y_d)$. Proof that (10) provides tracking with bounded error, as well as internal stability appears in [3]. If $\|(x_d^{(0,2)}, y_d^{(0,2)})\|_\infty$ is sufficiently small, then we may use other choices of v_e such as

$$v_e = -\beta_1(\dot{\alpha} - L_F \alpha_e) - \beta_0(\alpha - \alpha_e) \tag{12}$$

$$v_e = -\beta_1 \dot{\alpha} - \beta_0(\alpha - \alpha_e) \tag{13}$$

Simulations. Figures 3 and 5 show the results of two simulations. Choice (12) for v_e was used. Initial conditions in both cases were $y(0) = 5$[m], $\dot{x}(0) = 2.5$[m/s], $x(0) = \dot{y}(0) = \ddot{x}(0) = \ddot{y}(0) = 0$. The dynamic inverter gain μ was 10. The control gains were $\gamma_0 = 1$, $\gamma_1 = 3$, $\gamma_2 = 3$, $\beta_0 = 25$, $\beta_1 = 10$. In the first simulation $(x_d, y_d) = (5t, 0)$. The top graph of Figure 3 shows the resulting path in the plane (solid)

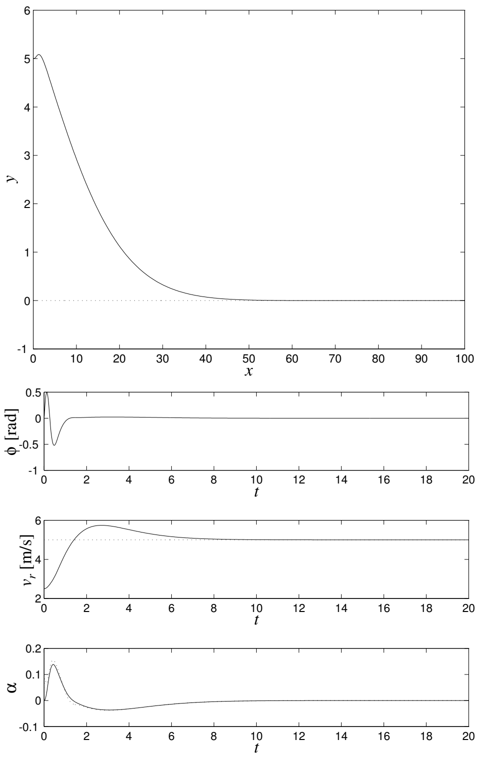

Figure 3: Target path $(x_d, y_d) = (5t, 0)$.

along with the desired path (dotted). The bottom three graphs show the steering angle ϕ, the rear-wheel velocity v_r (solid) and desired rear-wheel velocity (dotted), and (bottom) the roll-angle α (solid) with α_e (dotted). Figure 4 shows schematically the convergence of α to α_e and the resulting path. In the second simulation $(x_d, y_d) = (5t, 2\sin(0.2\pi t))$. Figure 5 shows the resulting path in the plane, steering ϕ, velocity v_r and with desired velocity, roll α and α_e.

3. References

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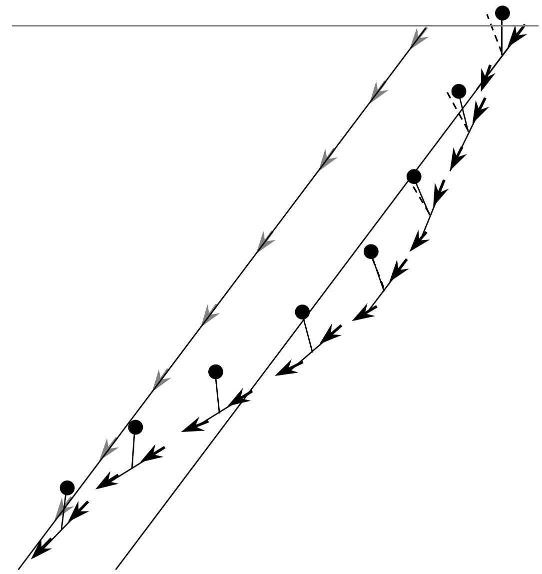

Figure 4: Attraction to α_e.

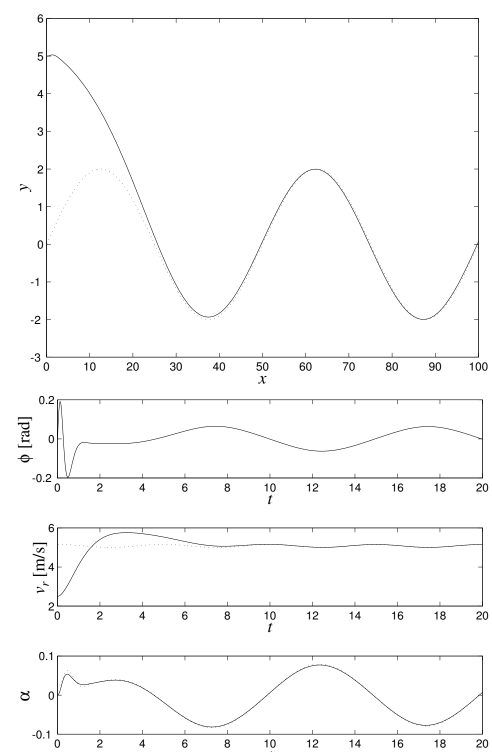

Figure 5: Target path $(x_d, y_d) = (5t, 2\sin(0.2\pi t))$.