Control of Nonholonomic Systems With Dynamically Decoupled Actuators

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Abstract

A compensator is derived for a class of nonholonomic systems incorporating both dynamically coupled and uncoupled actuators. The compensator affords exponential convergence to smooth trajectories in a reduced state space.

1 Introduction

In control engineering practice it is often the case that one uses actuator subsystems that are designed so that the internal dynamics of actuation are unaffected by the plant dynamics. Consider a three-wheel cart with two parallel rear wheels on a fixed axle, and a single front wheel of negligible moment of inertia on a vertical steering axis. In a dynamic model of the cart the steering angle does not appear as a generalized coordinate of the mechanical system, though it affects the system trajectory through the constraint that the wheels roll without slipping. In this paper the manner in which decoupled controls affect the dynamics of the plant through the nonholonomic constraints is studied. A tracking control law for smooth trajectories consistent with the constraints is then derived. The methodology presented here may be considered to be a modification of that established by Bloch, McClamroch, and Reyhanoglu [BRM91] [RMB93] for Caplygin systems.

2 Problem Description

Let the generalized coordinates $q \in \Re^n$ of a physical system be partitioned into $q = (q_1, q_2)$ with $q_1 \in \Re^{n-m}$ and $q_2 \in \Re^m$. Consider the three part system described by

$$\dot{x} = w \tag{1}$$

$$M(q_1)\,\ddot{q} = f(q_1, \dot{q}) + G(q_1)\,u + A^T(x, q_1)\,\lambda \tag{2}$$

$$A(x, q_1)\,\dot{q} = 0 \tag{3}$$

$$x,\ w \in \Re^k,\ \lambda \in \Re^m,\ u \in \Re^p,\ M \in \Re^{n\times n},\ G \in \Re^{n\times p}$$

Assume u is piecewise continuous, all other functions are smooth in their arguments, M is positive definite, and G is full rank. Equation (1) models the presence of dynamically decoupled actuators. Equation (2) models a plant subject to forces of constraint. Equation (3) models extrinsic nonholonomic constraints. A distinction may be drawn between plants with only intrinsic nonholonomic constraints, such as velocity constraints arising from conserved momenta, and those plants that are considered here.

Consider what will be called the *free problem*,

$$M(q_1)\,\ddot{q} = f(q_1, \dot{q}) + G(q_1)\,u \tag{4}$$

In physical systems that are nonholonomic by virtue of only intrinsic constraints there is no distinction between the free problem and the constrained problem. Note that (4) does not depend on the generalized coordinates q_k, but does depend on $\dot{q}_k$. By itself (4) defines the dynamics of a physical system not subject to the extrinsic nonholonomic constraints (3) (e.g., our cart sliding across ice), though it may be subject to exogenous inputs u as well as intrinsic nonholonomic constraints (e.g., conservation of linear and angular momentum of the sliding cart). In the system $\{(1),(4)\}$ the input w cannot affect the trajectory of q. When extrinsic constraints (3) are brought into the picture, the term $A^T(x, q_1)\lambda$ must be added to the formerly free problem (4) in order to account for changes in solution trajectories due to the extrinsic constraint forces. This brings the system to the form $\{(1), (2), (3)\}$ where the input w may affect the plant (2) through those states x on which the constraints (3) depend. The Lagrange multipliers λ can be determined by solving $\{(1), (2), (3)\}$ for a set of initial conditions. They may also be eliminated from $\{(1), (2), (3)\}$ as shown below.

Matrix A will be assumed to have the structure $A = [A_1 \,|\, A_2]$ where $A_2 \in \Re^{m\times m}$. Equation (3) may now be rephrased as

$$A_1(x, q_1)\,\dot{q}_1 + A_2(x, q_1)\,\dot{q}_2 = 0 \tag{5}$$

Since A_2 is invertible $\dot{q}_2$ may be expressed as a function of $\dot{q}_1$

$$\dot{q}_2 = -A_2(x, q_1)^{-1} A_1(x, q_1)\,\dot{q}_1 \tag{6}$$

Define the velocity map V which maps $\dot{q}_1$ to $\dot{q}$ through

$$V(x, q_1) = \begin{bmatrix} I \\ -A_2(x, q_1)^{-1} A_1(x, q_1) \end{bmatrix} \tag{7}$$

where $I \in R^{(m-n)\times(m-n)}$ denotes the identity matrix. Note that $A(x,q_1)V(x,q_1) = 0$. The velocity constraints (3) may now be expressed as

$$\dot{q} = V(x,q_1)\dot{q}_1 \tag{8}$$

where $\dot{q}_1$ is unconstrained when equation (3) is considered in isolation from (1) and (2). Through (8) system $\{(1), (2), (3)\}$ takes the form

$$\begin{aligned} \dot{x} &= w \\ \dot{q} &= V(x,q_1)\dot{q}_1 \\ M(q_1)\ddot{q} &= f(q_1,\dot{q}) + G(q_1)u + A^T(x,q_1)\lambda \end{aligned} \tag{9}$$

Equations (9) incorporate our assumptions thus far and define the class of systems in which we are interested. A control law may now be derived that will cause $q_1(t)$ to exponentially converge to and track a desired smooth trajectory. Once q_1 can be made to exponentially converge to any desired smooth trajectory, then, if $A_1 \neq 0$, some subset of $\{q_{2,i}\}_{i\in\underline{m}}$ can be made to exponentially converge to arbitrary and desired smooth trajectories, respecting, of course, the domain of our assumptions.

3 Derivation of a Control Law

Equation (2), consisting of n second-order differential equations, may now be reduced to a system of $n-m$ second order differential equations in q_1 by incorporating the nonholonomic constraints (3) into equation (2). Note that once q_1 is obtained, the m first order equations (6) may be integrated to obtain q_2.

For positive integer n let $\underline{n} := \{1, 2, \ldots, n\}$. Differentiating the terms of the left hand side of (5) with respect to time yields

$$\begin{aligned} \frac{d}{dt}(A_1(x,q_1)\dot{q}_1) &= \left[\frac{d}{dt}\sum_j A_{1,i,j}(x,q_1)\dot{q}_{1,j}\right]_{i\in\underline{m}} \\ &= A_1(x,q_1)\ddot{q}_1 + \left[\sum_{k,j}\frac{\partial A_{1,i,j}(x,q_1)}{\partial q_{1,k}}\dot{q}_{1,j}\dot{q}_{1,k} + \sum_{k,j}\frac{\partial A_{1,i,j}(x,q_1)}{\partial x_k}\dot{q}_{1,j}\dot{x}_k\right]_{i\in\underline{m}} \\ &= A_1(x,q_1)\ddot{q}_1 + K_1(x,q_1,\dot{q}_1)\dot{q}_1 + H_1(x,q_1,\dot{q}_1)w \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{d}{dt}(A_2(x,q_1)\dot{q}_2) &= \left[\frac{d}{dt}\sum_j A_{2,i,j}(x,q_1)\dot{q}_{2,j}\right]_{i\in\underline{m}} \\ &= A_2(x,q_1)\ddot{q}_2 + \left[\sum_{k,j}\frac{\partial A_{2,i,j}(x,q_1)}{\partial q_{1,k}}\dot{q}_{2,j}\dot{q}_{1,k} + \sum_{k,j}\frac{\partial A_{2,i,j}(x,q_1)}{\partial x_k}\dot{q}_{2,j}\dot{x}_k\right]_{i\in\underline{m}} \end{aligned}$$

$$= A_2(x,q_1)\,\ddot{q}_2 + K_2(x,q_1,\dot{q}_1)\,\dot{q}_1 + H_2(x,q_1,\dot{q}_1)\,w$$

where we have eliminated $\dot{q}_k$ by using (6), and $K_1 \in R^{m\times(n-m)}$, $H_1 \in R^{m\times k}$, $K_2 \in R^{m\times(n-m)}$, $H_2 \in R^{m\times k}$, are defined in the obvious manner. Combining (5) and (11) we obtain an expression for $\ddot{q}_2$ in terms of $\ddot{q}_1$, $\dot{q}_1$, and w. Now $\ddot{q}$ may be written as

$$\begin{aligned}
\ddot{q} &= \begin{bmatrix} I \\ -A_2^{-1}(x,q_1)\,A_1(x,q_1) \end{bmatrix} \ddot{q}_1 \\
&\quad + \begin{bmatrix} 0 \\ -A_2^{-1}(x,q_1)\,(K_1(x,q_1,\dot{q}_1) + K_2(q_1,\dot{q}_1)) \end{bmatrix} \dot{q}_1 \\
&\quad + \begin{bmatrix} 0 \\ -A_2^{-1}(x,q_1)\,(H_1(x,q_1,\dot{q}_1) + H_2(x,q_1,\dot{q}_1)) \end{bmatrix} w \\
&= V(x,q_1)\,\ddot{q}_1 + K(x,q_1,\dot{q}_1)\,\dot{q}_1 + H(x,q_1,\dot{q}_1)\,w \qquad (11)
\end{aligned}$$

where $K \in \Re^{n\times(n-m)}$ and $H \in R^{n\times k}$ are the coefficient matrices for $\dot{q}_1$ and w, respectively.Substituting (11) into (2) gives

$$\begin{aligned}
&M(q_1)\,V(x,q_1)\,\ddot{q}_1 \\
&= f(q_1,\dot{q}) - M(q_1)\,K(x,q_1,\dot{q}_1)\,\dot{q}_1 + A^T(x,q_1)\,\lambda + G(q_1)\,u \\
&\quad - M(q_1)\,H(x,q_1,\dot{q}_1)\,w \qquad (12)
\end{aligned}$$

where the input w has become an explicit part of the equations of motion of the plant. Left multiplying (12) by $V(x,q_1)^T$ yields the *reduced* dynamical equations

$$V^T(x,q_1)\,M(q_1)\,V(x,q_1)\,\ddot{q}_1 = F(x,q_1,\dot{q}_1) + B(x,q_1,\dot{q}_1)\begin{bmatrix} u \\ w \end{bmatrix} \qquad (13)$$

where

$$\begin{aligned}
B(x,q_1,\dot{q}_1) &:= V^T(x,q_1)\,[\,G(q_1) \quad | \quad -M(q_1)\,H(x,q_1,\dot{q}_1)\,] \\
F(x,q_1,\dot{q}_1) &:= V^T(x,q_1)\,(f(q_1, V(x,q_1)\,\dot{q}_1) - M(q_1)\,K(x,q_1,\dot{q}_1)\,\dot{q}_1)
\end{aligned}$$

and the fact that $V^T A^T = 0$ has been used. Since V is of full rank $V^T M V$ is positive definite. The Lagrange multipliers have been eliminated from the n dynamical equations (2) and reduced them to a system of $n-m$ second order differential equations.

Fact 1 *Assume that B is of rank $n-m$. Then B has a right-inverse $B^\dagger := B^T(BB^T)^{-1}$. Let $C_1, C_0 \in \Re^{(n-m)\times(n-m)}$ be constant, positive-definite matrices. Under these assumptions, and the assumption that the triple $(x,\ q_1,\ q_2)$ remains in an open domain in which our assumptions are valid, the trajectories $q_1(t)$ can be made to locally exponentially converge to any desired smooth trajectory $q_{1d}(t)$ by the state feedback*

$$\begin{bmatrix} u \\ w \end{bmatrix} = B^\dagger\left(V^T M V\,(\ddot{q}_{1d} - C_1(\dot{q}_1 - \dot{q}_{1d}) - C_0(q_1 - q_{1d})) - F\right) \qquad (14)$$

□

Note that if the rank of B is less than $n - m$ it may still be possible to achieve trajectory tracking of smooth functions of $q_1(t)$ through more involved feedback linearization [Isi89].

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