An Internal Equilibrium Manifold Method of Tracking for Nonlinear Nonminimum Phase Systems

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American Control Conference, 21-23 June 1995, Seattle

Abstract

A new nonlinear controller for nonlinear nonminimum phase systems is described. The design is based upon an "internal equilibrium manifold," a submanifold embedded in the state space. By controlling the state of the system to remain in a neighborhood of the submanifold, tracking with bounded error is achieved for an open set of output trajectories. The controller provides a significantly extended region of attraction over linear regulators. Regulation and tracking for the inverted pendulum on a cart demonstrates the control scheme.

1 Problem Statement

Let $\underline{k} := \{1, \ldots, k\}$ for any integer $k > 0$. For the purposes of this paper we will consider the single-input, single-output, n-dimensional nonlinear control systems of the form

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}_i &=& x_{i+1}, \quad i \in \underline{m-1} \\ \dot{x}_m &=& u \\ \dot{\alpha}_i &=& \alpha_{i+1}, \quad i \in \underline{p-1} \\ \dot{\alpha}_p &=& f(x,\alpha) + g(x,\alpha)u \\ \\ y &=& x_1 \end{array} \right. \tag{1}$$

with input $u \in \mathbb{R}$, output $y \in \mathbb{R}$, $x := (x_1, \ldots, x_m)$, $\alpha := (\alpha_1, \ldots \alpha_p)$, and $n = m+p$. The coordinates (x, α) are assumed to be defined on an open neighborhood $\mathcal{X} \subset \mathbb{R}^n$ of the origin. It will be assumed that (1) has an isolated equilibrium at the origin $(x, \alpha) = (0, 0)$. The functions $f(x, \alpha)$ and $g(x, \alpha)$ are assumed to be smooth in their arguments. Furthermore, $g(x, \alpha)$ is assumed to be non-zero for all (x, α) in $\mathcal{X}$.

It will be our objective to cause the output $y(t)$ to track a desired reference signal $y_d(t)$ which, for convenience later, we will consider to be the output of the system

$$\Sigma_d \left\{ \begin{array}{rcl} \dot{w}_i &=& w_{i+1}, \quad i \in \underline{n-1} \\ \dot{w}_n &=& y_d^{(n)}(t) \\ \\ y_d &=& w_1. \end{array} \right. \tag{2}$$

Note that the dimension of Σ_d is n, so we require $y_d(t)$ to be C^n in t. In referring to Σ_d we will always assume that $w_i(0)$, $i \in \underline{n}$ have been set so that the output of Σ_d is the desired output. Thus $w_i \equiv y_d^{(i-1)}$, for $i \in \underline{n}$. We will also associate with all possible y_d the norm

$$\|y_d\|_Y = \sup_t(|y_d(t)|, |y_d^{(1)}(t)|, \ldots, |y_d^{(n)}(t)|). \tag{3}$$

When we refer to y_d being drawn from an open set, we will mean an open set induced by the norm $\|\cdot\|_Y$.

Let $L_f h := dh \cdot f$ denote the Lie derivative of a function h along the vector field f. Nonlinear control systems of the form

$$\begin{array}{rcl} \dot{\bar{x}} &=& \bar{f}(\bar{x}) + \bar{g}(\bar{x})u \\ \bar{y} &=& h(\bar{x}) \end{array} \tag{4}$$

may be brought into the form (1) in a number of ways. For instance if the output $\bar{y}$ has relative degree m (see [1], Chapter 4, for the definition and uses of relative degree.) and there exists another function $\phi(\bar{x})$ having relative degree p and such that

$$\Phi(\bar{x}) := [L_{\bar{f}}^0 h, \ldots, L_{\bar{f}}^{m-1} h, L_{\bar{f}}^0 \phi, \ldots, L_{\bar{f}}^{p-1} \phi](\bar{x}) \tag{5}$$

is a diffeomorphism, then the coordinate transform $\bar{x} \mapsto \Phi(\bar{x})$ brings (4) to the form of Σ.

Now consider the system

$$\left\{ \begin{array}{rcl} \dot{\alpha}_i &=& \alpha_{i+1}, \quad i \in \underline{p-1} \\ \dot{\alpha}_p &=& f(0, \alpha) \end{array} \right. \tag{6}$$

obtained by restricting the input u and the output y of Σ to be identically zero. The class of systems Σ that we will be interested in are characterized by the property that $\alpha = 0$ is an *unstable* equilibrium of (6). The dynamics (6) are referred to as the *internal dynamics* of Σ, and α will be referred to as the *internal state*. Systems with unstable internal dynamics are referred to as being *nonminimum phase* systems due to their correspondence to linear systems having transmission zeros in the right half

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of the complex plane. Their existence provokes the study of an important problem for nonlinear control; regulation and output tracking with internal stability. Unlike systems with stable internal dynamics, non-minimum phase systems with given initial conditions cannot be made to exactly track arbitrary elements of an open set of output reference trajectories while maintaining bounded internal dynamics. Instead we will offer a feedback controller for Σ that affords *approximate* tracking of reference trajectories $t \mapsto y_d(t)$ drawn from an open set, while maintaining bounded and well-behaved internal state.

It should be noted that single-input single-output systems of the form Σ can be brought to a normal form [1] in which u does not appear in the internal dynamics. This is a consequence of the fact that the input distribution associated with Σ is one-dimensional and therefore involutive. For higher dimensional systems, however, involutivity of the input distribution, is exceedingly rare. We will rely upon the appearance of u in the internal dynamics. This allows the control method which we will present here to be easily extended to the multi-input multi-output case without reliance upon involutivity of the input distribution [2]. If the input distribution is indeed involutive, our method is unaffected by this.

In Section 2 we define the internal equilibrium manifold and construct a controller that makes a neighborhood of this manifold attractive and invariant, thereby providing approximate tracking with bounded and well-behaved internal dynamics. In Section 3 we illustrate both the problem of interest and our solution by applying our controller to a model of an inverted pendulum on a cart, where the cart position is the output we wish to cause to track a desired trajectory. We close with a brief discussion of some features of our results.

2 Output Tracking

In this section we will describe a controller that provides the desired output tracking behavior along with bounded and well behaved internal dynamics. For a more formal presentation of the methods and arguments described here see [2].

Consider Σ with its internal dynamics stripped away. We call the resulting system the *external dynamics* Σ_{ext},

$$\Sigma_{\text{ext}} \left\{ \begin{array}{rcl} \dot{x}_i & = & x_{i+1}, \quad i \in \underline{m-1} \\ \dot{x}_m & = & u \\ & & \\ y & = & x_1 \end{array} \right. . \tag{7}$$

Define a *nominal* input u_y by

$$u_y(w,x) := w_m - \sum_{i=1}^{m-1} \gamma_i (x_i - w_i) \tag{8}$$

where γ_i, $i \in \underline{m-1}$, are real, and w_i are the state variables of Σ_d. Application of u_y to Σ_{ext} causes $y(t)$ to converge to $y_d(t)$ exponentially as long as the γ_i are chosen to be the coefficients of a characteristic polynomial $r^m + \gamma_m r^{m-1} + \cdots + \gamma_1$ whose roots all have negative real parts. Application of $u_y(w,x)$ to Σ_d results in exponentially stable tracking.

Define the *complementary output* $\bar{y} = \alpha_1$. This output may be considered to correspond to the function ϕ mentioned in the last section, i.e. $\bar{y} = \phi(\bar{x})$, in the context of bringing a control system to the form Σ. Note that $\bar{y} \equiv 0$ and $u \equiv 0$ implies $\alpha \equiv 0$. Consider also a reference signal $\bar{y}_d(t)$ that is C^p in t. Consider the input transformation

$$u = g^{-1}(x,\alpha)\left(-f(x,\alpha) + \bar{v}\right). \tag{9}$$

Application of this transformation to Σ gives the system

$$\bar{\Sigma} \left\{ \begin{array}{rcl} \dot{x}_i & = & x_{i+1}, \quad i \in \underline{m-1} \\ \dot{x}_m & = & g^{-1}(x,\alpha)\left(-f(x,\alpha) + \bar{v}\right) \\ \dot{\alpha}_i & = & \alpha_{i+1}, \quad i \in \underline{p-1} \\ \dot{\alpha}_p & = & \bar{v} \\ & & \\ \bar{y} & = & \alpha_1 \end{array} \right. . \tag{10}$$

This system $\bar{\Sigma}$ has a similar and complementary structure to Σ since, regarding $\bar{y}$ as an output and $\bar{v}$ as an input, x is now the internal state of $\bar{\Sigma}$ and α is now the external state. The internal dynamics of $\bar{\Sigma}$ may or may not be stable depending upon the structure of $g(x,\alpha)$ and $f(x,\alpha)$. By choosing

$$\bar{v} = \bar{y}_d^{(p)}(t) - \sum_{i=1}^{p-1} \beta_i (\alpha_i - \bar{y}_d^{(i-1)}) \tag{11}$$

for β_i chosen to be the coefficients of a polynomial $r^p + \beta_p r^{p-1} + \cdots + \beta_1$ having roots with strictly negative real parts, we can cause the output $\bar{y}(t)$ of $\bar{\Sigma}_{\text{ext}}$, the external dynamics of $\bar{\Sigma}$, to exponentially converge to any desired trajectory $\bar{y}_d(t)$ in an open set. We summarize this important structural feature of Σ by noting the following — We can make $\bar{y} = \alpha_1$ track what we want it to track, and we can make $y = x_1$ track what we want it to track. We cannot, however, make both y and $\bar{y}$ track two arbitrary trajectories simultaneously.

Given our problem, and the structure of Σ, a logical question to ask is then, what bounded trajectory can we make $\bar{y}$ track (approximately) that will cause y to converge (approximately) to y_d. Our control law is, in an approximate sense, an answer to this question.

Consider now the set of algebraic equations

$$\left\{ \begin{array}{rcl} \epsilon_i & = & \alpha_{i+1}, \quad i \in \underline{p-1} \\ \epsilon_p & = & f(x,\alpha) + g(x,\alpha)u \end{array} \right. \tag{12}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_p) \in \mathbb{R}^p$ is small. We will regard the p equations (12) as describing an x-dependent relation between α_1 (the only α_i not constrained to be ϵ_{i-1}) and u.

Suppose $\epsilon = 0$ in (12). Assume that for all x and α_1 in a neighborhood of zero,

$$\frac{\partial (g^{-1} f)(x; \alpha_1, 0, \ldots, 0)}{\partial \alpha_1} \neq 0. \tag{13}$$

By our assumptions and the implicit function theorem, equations (12) determine what we will regard as an x-dependent diffeomorphism from u to $\alpha_1(u)$. By replacing u by $\bar{u}(w,x)$ in (12) we obtain $\alpha_e(w,x) = \alpha_1(\bar{u}(w,x))$, a w-dependent mapping from the external state-space to $\mathbb{R}$. Note that $\alpha_e(w,x)$ is *defined* by setting $\epsilon = 0$. If $\epsilon \neq 0$, $\alpha_e(w,x)$ remains the same as when $\epsilon = 0$.

Definition 2.1 The *internal equilibrium manifold* of Σ is the set

$$\mathcal{E} = \{(x,\alpha) \in \mathbb{R}^n \mid \alpha_1 = \alpha_e(w,x), \alpha_i = 0,\ i \in \{2,\ldots,p\}\}. \tag{14}$$

The manifold $\mathcal{E}$ is an m-dimensional and w-dependent regular submanifold which we may regard as being embedded in the state-space $\mathbb{R}^n$. We may also regard $\mathcal{E}$ as a w-dependent *graph* over the external state-space $\mathbb{R}^m$, embedded in a state space $\mathbb{R}^{m+1}$. Since α_e is smooth, and $\alpha_e(0,0) = 0$ we know that $\mathcal{E}$ is bounded on a neighborhood of the origin $(x,\alpha) = (0,0)$ for $\|y_d\|_Y$ (and hence $\|w\|_\infty$) sufficiently small. Also note that by our smoothness assumptions, and for a suitable restriction to a neighborhood of the the origin of $\mathbb{R}^n$, if ϵ is allowed to stray from zero somewhat, the resulting manifold defined by (12) varies from $\mathcal{E}$ only slightly.

Regarding $\mathcal{E}$ as a w-dependent graph over the external state space $\mathbb{R}^m$, consider that every trajectory $x(t)$ in the external state-space has an image in $\mathcal{E}$. Moreover, since $\mathcal{E}$ is bounded, this image is bounded. Define $F_d := [w_2,\ldots,w_n,y_d^{(n)}(t)]^T$, $F_y(u) := [x_2,\ldots,x_m,u]^T$, and let $F(u) := [F_d^T, F_y(u)^T]^T$. The vector field $F(u_y(w,x))$, for instance, governs the dynamics of $\{\Sigma_d, \Sigma_{\text{ext}}\}$ when the input u_y is applied.

Our strategy will be as follows: We will choose a feedback $\hat{u}(w,y_d^{(n)},x,\alpha)$ that makes *a neighborhood of* $\mathcal{E}$ attractive and invariant. Once α_1 is close to $\alpha_e(w,x)$ for some x, we will make $\bar{y} = \alpha_1$ approximately track the image of x in $\mathcal{E}$. If $\|y_d\|_Y$ is sufficiently small, and α_1 is sufficiently close to $\alpha_e(w,x)$ then $\hat{u}$ will be close to $u_y(w,x)$ and therefore $F(\hat{u})$ will be close to $F(u_y)$. In order to make $\bar{y}$ approximately track the image of x we must estimate time derivatives of α_e. Since $F(u_y)$ is close to $F(\hat{u})$ we will use $F(u_y)$ to estimate the time derivatives of α_e. We can't use $F(\hat{u})$ since it is $\hat{u}$ that we are trying to determine. When $F(\hat{u}) \approx F(u_y)$, x flows to a neighborhood of $(w_1,\ldots,w_m)$ providing approximate tracking. If $y_d \equiv 0$ we must assure ourselves that the coefficients of our control law are such that the Σ is stable at the origin, since convergence to a neighborhood of an equilibrium does not imply stability of the equilibrium. Therefore we require local controllability at the origin.

Our choice of state feedback is

$$\begin{aligned} \hat{u}(w,y_d^{(n)},x,\alpha) &= g(x,\alpha)^{-1}(-f(x,\alpha)+v) \\ v &:= L_{\tilde{F}}^p\alpha_e - \textstyle\sum_{i=1}^p \beta_i(\alpha_i - L_{\tilde{F}}^{i-1}\alpha_e) \end{aligned} \tag{15}$$

with β_i specified as above. The dependence of $\hat{u}$ on $y_d^{(n)}$ is through the term $L_{\tilde{F}}^p\alpha_e$. This is why we require $y_d(t)$ to be C^n.

Recall that

$$\begin{aligned} 0 \;=\; & f(x;\alpha_e(w,x),0,\ldots,0) \\ & + g(x;\alpha_e(w,x),0,\ldots,0)u_y(w,x). \end{aligned} \tag{16}$$

Consequently

$$-(g^{-1}f)(x;\alpha_e(w,x),0,\ldots,0) = u_y(w,x). \tag{17}$$

Expanding $-(g^{-1}f)(x,\alpha)$ about $\alpha = (\alpha_e,0,\ldots,0)$ then gives

$$\begin{aligned} & -(g^{-1}f)(x,\alpha) \\ & = u_y(w,x) - \textstyle\sum_{i=2}^p \frac{\partial}{\partial\alpha_i}(g^{-1}f)(x;\alpha_e,0,\ldots,0)\alpha_i \\ & = u_y(w,x) + O(|\alpha_2|,\ldots,|\alpha_p|). \end{aligned} \tag{18}$$

Then

$$\hat{u} \;=\; u_y(w,x) + O(|\alpha_2|,\ldots,|\alpha_p|) + g^{-1}(x,\alpha)v. \tag{19}$$

It will also be convenient to define error coordinates $z_i = x_i - w_i$ and $e_i = \alpha_i - \alpha_e^{(i-1)}$. Then v takes the form

$$\begin{aligned} v \;=\; & \alpha_e^{(p)} - \textstyle\sum_{i=1}^p \beta_i(\alpha_i - \alpha_e^{(i-1)}) + (L_{\tilde{F}}^p\alpha_e - \alpha_e^{(p)}) \\ & + \textstyle\sum_{i=1}^p \beta_i(L_{\tilde{F}}^{i-1}\alpha_e - \alpha_e^{(i-1)}) \\ =\; & \alpha_e^{(p)} - \textstyle\sum_{i=1}^p \beta_i z_i + (L_{\tilde{F}}^p\alpha_e - \alpha_e^{(p)}) \\ & + \textstyle\sum_{i=1}^p \beta_i(L_{\tilde{F}}^{i-1}\alpha_e - \alpha_e^{(i-1)}) \end{aligned} \tag{20}$$

and u_y takes the form

$$u_y = w_{m+1} - \sum_{i=1}^m \gamma_i e_i. \tag{21}$$

Substituting the above results into Σ with input $\hat{u}$ gives

$$\Sigma \left\{ \begin{aligned} \dot{e}_i &= e_{i+1},\ \ i \in \underline{m-1} \\ \dot{e}_m &= -\textstyle\sum_{i=1}^m \gamma_i e_i + O(|\alpha_1|,\ldots,|\alpha_p|) + g^{-1}v \\ \dot{z}_i &= z_{i+1},\ \ i \in \underline{p-1} \\ \dot{z}_p &= -\textstyle\sum_{i=1}^p \beta_i z_i + (L_{\tilde{F}}^p\alpha_e - \alpha_e^{(p)}) \\ & \quad -\textstyle\sum_{i=2}^p \beta_i(L_{\tilde{F}}^{i-1}\alpha_e - \alpha_e^{(i-1)}) \\ \\ y &= y_d + e_1 \end{aligned} \right. \tag{22}$$

Consider some structural features of (22). The z-dynamics are the dynamics of an exponentially stable linear system with a nonlinear non-vanishing perturbation. For suitable bounds on the perturbation one may guarantee convergence of z to a neighborhood of the origin. We have a similar story for the e-dynamics, also regarded as a stable linear system with a nonlinear nonvanishing perturbation. Thus, under suitable conditions on the coefficients γ_i, β_i, and the norms of the perturbations we achieve convergence of (z,e) to a neighborhood of the origin. This convergence implies approximate tracking with bounded and well-behaved internal dynamics. It may be verified that the non-vanishing part of the nonlinear perturbation in (22) is due to $y_d^{(n)}$. Thus, only for $y_d^{(n)} = 0$ can our scheme achieve asymptotically exact tracking.

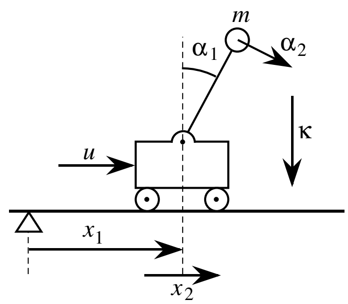

Figure 1: Inverted pendulum on a cart.

3 An Example

The classical control experiment of the inverted pendulum on a cart (see Figure 1) will be used to illustrate both the problem in which we are interested as well as application of the control technique we have described.

The position of the cart is parameterized by $x_1 \in \mathbb{R}$, the linear velocity of the pendulum pivot by $x_2 \in \mathbb{R}$, the angle of the pendulum away from upright by $\alpha_1 \in (-\pi/2,\ \pi/2) \subset S^1$, and the and the angular velocity of the pendulum by $\alpha_2 \in \mathbb{R}$. We will assume that sufficient force is available so that we may consider the *acceleration* of the cart to be the input u to our system. This assumption is equivalent to feedback linearization via differentiation of the output $y = x_1$ with a change of input coordinates and brings our control system to the standard form of Σ,

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& u \\ \dot{\alpha}_1 &=& \alpha_2 \\ \dot{\alpha}_2 &=& \kappa \sin(\alpha_1) - \cos(\alpha_1) u \end{array} \right. \tag{23}$$

where, for convenience, we have taken the pendulum length $l = 1$. The gravitational acceleration is κ.

For this problem Σ_d is of dimension $n = 4$ with $w_1(0) = y_d(0)$, $w_2(0) = y_d^{(1)}(0)$, $w_3(0) = y_d^{(2)}(0)$, and $w_4(0) = y_d^{(3)}(0)$.

Let

$$u_y(w, x) := w_3 - \beta_2(x_2 - w_2) - \beta_1(x_1 - w_1) \tag{24}$$

where β_1 and β_2 are real numbers chosen such that $r^2 + \beta_2 r + \beta_1 = 0$ has roots with strictly negative real parts.

For this problem we have

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \\ \dot{w}_4 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ w_3 \\ w_4 \\ y_d^{(4)} \\ x_2 \\ w_3 - \beta_2(x_2 - w_2) - \beta_1(x_1 - w_1) \end{bmatrix} =: F(u_y) \tag{25}$$

The internal equilibrium manifold for the pendulum is the set $\mathcal{E}$ of all (e, α) satisfying

$$\begin{array}{rcl} 0 &=& \alpha_2 \\ 0 &=& \kappa \sin(\alpha_1) - \cos(\alpha_1) u_y(w, x). \end{array} \tag{26}$$

Note also that in this case, due to the fact that the constraint equations (26) depend on x only through $u_y(w, x)$, we could pose $\mathcal{E}$ as a fixed graph over the space $\mathbb{R}$ in which u_y resides; i.e. the equations (26) describe a relation between α and u which does not depend on x or t. For generality we will ignore this, though our observation is reflected in the property that the level sets of $\mathcal{E}$ for our chosen u_y defined by (24) will be parallel lines.

Call the solution of the second equation of (26), α_e. In the present case we may determine α_e explicitly as $\alpha_e = \tan^{-1}(u_y/\kappa)$.

Again, let $\tilde{F} := F(u_y(w, x))$. For our final control law we will need $L_{\tilde{F}}\alpha_e$, and $L^2_{\tilde{F}}\alpha_e$. We obtain these by differentiating the second equation of (26) along $\tilde{F}$, and then applying some algebraic manipulation to obtain

$$\begin{array}{rcl} L_{\tilde{F}}\alpha_e &=& (\kappa \cos(\alpha_e) + \sin(\alpha_e) u_y)^{-1} \cos(\alpha_e) L_{\tilde{F}} u_y \\ L^2_{\tilde{F}}\alpha_e &=& L_{\tilde{F}}\alpha_e \left(\frac{L^2_{\tilde{F}} u_y}{L_{\tilde{F}} u_y} - 2 \frac{u_y}{\kappa} L_{\tilde{F}}\alpha_e \right) \end{array} \tag{27}$$

where

$$\begin{array}{rcl} u_y &=& w_3 - \beta_2(x_2 - w_2) - \beta_1(x_1 - w_1) \\ L_{\tilde{F}} u_y &=& w_4 - \beta_2(u_y - w_3) - \beta_1(x_2 - w_2) \\ L^2_{\tilde{F}} u_y &=& y_d^{(4)} - \beta_2(L_{\tilde{F}} u_y - w_4) - \beta_1(u_y - w_3) \end{array} \tag{28}$$

The tracking control law for the pendulum is then

$$\left\{ \begin{array}{rcl} \hat{u} &=& -(-\kappa \sin(\alpha_1) + v) / \cos(\alpha_1) \\ v &=& L^2_{\tilde{F}}\alpha_e - \gamma_2(\alpha_2 - L_{\tilde{F}}\alpha_e) - \gamma_1(\alpha_1 - \alpha_e) \end{array} \right. \tag{29}$$

Again, adjustment of the β_i's and γ_i's must be made so that linearization of Σ at the origin is stable when $u = \hat{u}$ and $y_d \equiv 0$.

A simulation shows the results first for the problem of regulating the cart and pendulum to the origin, and second, the problem of tracking a sinusoidal trajectory. For the regulation simulation we have started the pendulum at a large angle in order to demonstrate the size of the domain of attraction of the regulation. Values of $\beta_2 = 2$, $\beta_1 = 1$, were chosen for nominal critical damping of the output error. Gains $\gamma_2 = 7.04$, and $\gamma_1 = 12.39$ were then used to provide both convergence of α to α_e and a stable linearization at the origin. For this simulation the initial conditions were $x_1(0) = 1$, $x_2(0) = 0$, $\alpha_1(0) = 3\pi/8$, and $\alpha_1(0) = 0$. Figure 2 shows the value of the output $y = x_1(t)$ in the top graph, and the value of $\alpha_1(t)$ and well as $\alpha_e(t)$, shown dotted, in the bottom graph. We have achieved similar results for a simulation with parameters and initial conditions the same as above except for $\alpha_1(0) = 89^o$.

In the next simulation we apply our controller to the problem of tracking a sinusoidal cart trajectory $y_d(t) = 10\sin(2\pi 0.1 t)$.. The gains β_1, β_2, γ_1, and γ_2 were kept the

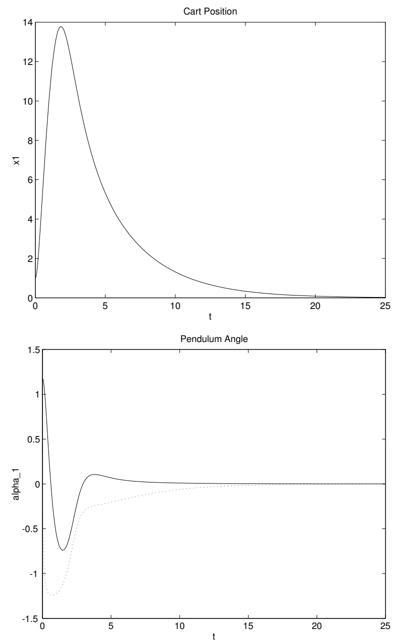

Figure 2: Regulation simulation results.

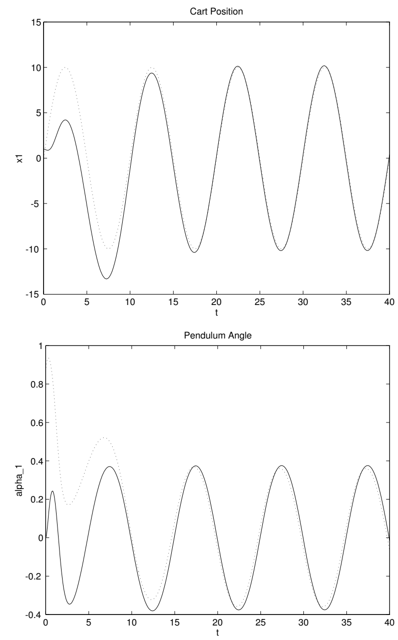

Figure 3: Tracking simulation results.

same as the regulation simulation. The initial conditions were $x_1(0) = 1$, $x_2(0) = 0$, $\alpha_1(0) = 0$, and $\alpha_1(0) = 0$. The top graph of Figure 3 shows the cart position (solid) and desired cart position (dotted). The bottom graph shows the pendulum angle $\alpha_1(t)$ (solid) along with $\alpha_e(t)$ (dotted).

4 Discussion and Conclusions

In general α_e is defined implicitly by a set of algebraic equations. In the case of the inverted pendulum we could solve this set of equations explicitly to obtain a "closed form" expression for α_e. In most cases, however, such explicit expressions cannot be found. In such cases one may apply a dynamic method for the inversion of nonlinear maps, called *dynamic inversion* [3][2] in order to produce an explicit representation for α_e as well as its Lie derivatives. The introduction of dynamic inversion adds dynamics to the controller but in such a way that does not destroy the stability and convergence properties of the control system.

We have only given a sketch of our control methodology. For a more rigorous handling and more examples with and without the use of dynamic inversion see [2]. For an application to planar trajectory tracking for a nonlinear nonholonmic model of a bicycle using dynamic inversion see [4].

Our control system may also be modified by setting the higher order Lie derivatives in the control law to zero. Performance suffers as a result, but for systems in which high performance is not required the computational load may be reduced by this modification.

For an up to date history of the problem of output tracking for nonlinear systems see [2], or the recent paper [5] which also contains an interesting approach to trajectory generation for nonminimum phase systems.

Conclusions: We have constructed an internal equilibrium manifold and a controller that pushes the state of a nonlinear non-minimum phase system toward that manifold. This has afforded approximate output tracking for nonlinear non-minimum phase systems while maintaining internal stability.

References

[1] A. Isidori, *Nonlinear Control Systems, An Introduction.* New York: Springer-Verlag, second ed., 1989.

[2] N. H. Getz, *Dynamic Inversion and the Control of Nonlinear Nonminimum Phase Systems.* PhD thesis, University of California at Berkeley, 1995.

[3] N. H. Getz and J. E. Marsden, "Dynamic inversion of nonlinear maps," Tech. Rep. 621, Center for Pure and Applied Mathematics, Berkeley, California, 19 December 1994. Submitted to IEEE Transactions on Automatic Control.

[4] N. H. Getz and J. E. Marsden, "Control for an autonomous bicycle," in *IEEE International Conference on Robotics and Automation*, (Nagoya, Aichi, Japan), IEEE, 21-27 May 1995.

[5] S. Devasia and B. Paden, "Exact output tracking for nonlinear time-varying systems," in *Proceedings of the 33rd IEEE Conference on Decision and Control*, vol. 3, (Lake Buena Vista), pp. 2346–2355, IEEE, December 1994.