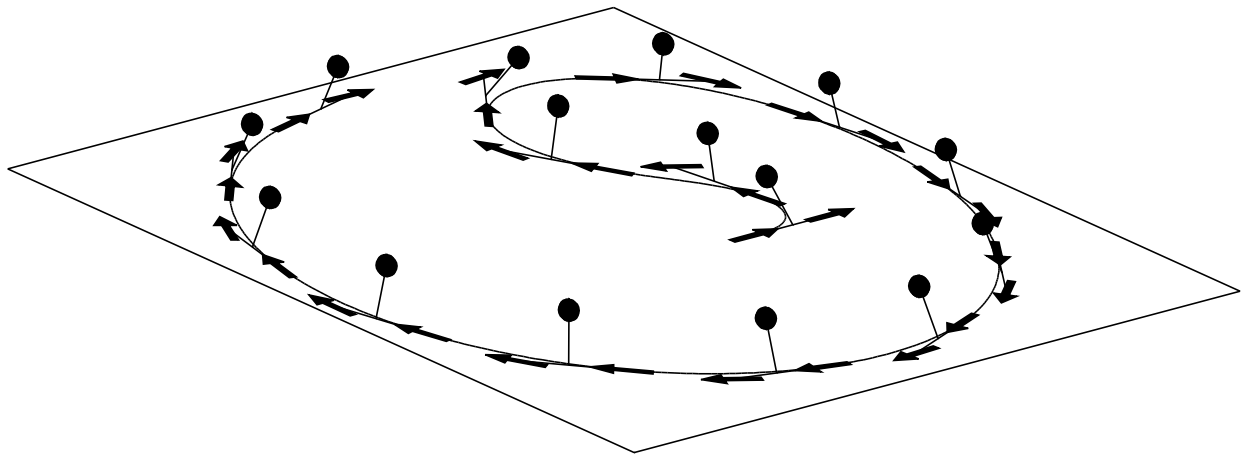


CONTROL OF BALANCE FOR A NONLINEAR
NONHOLONOMIC NON-MINIMUM PHASE MODEL
OF A BICYCLE



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Abstract

A feedback control law is derived that causes a nonlinear, nonholonomic, non-minimum phase model of a riderless powered two-wheeled bicycle to stably track arbitrary smooth trajectories of roll-angle and non-zero rear-wheel velocity.

1 Introduction

Though two-wheeled cycles comprise the second most common mode of transportation in the world, next only to walking, little is currently understood about their dynamics and even less about their control. This is due in part to the complexity of extant models (see, for instance [1], [2], [3]) and in part to the fact that the development of tools for the control of (inherently nonlinear) nonholonomic systems is of fairly recent vintage. Neimark and Fufaev [4] have studied the dynamics and stability of a simple nonlinear nonholonomic model of the bicycle similar to the vehicle modelled here, but their control analysis is confined to a linear controller, and they analyze stability of the control only for rectilinear constant-speed motion using Jacobian-linearization. In contrast the control scheme derived here is limited to neither rectilinear motion, nor constant speed.

More recent work in control of nonholonomic systems has been restricted, for the most part, to open loop control of drift-free vehicle models. The time-parameterization of the configuration space trajectories and controls for such models is arbitrary, affording a great deal of liberty in the planning of paths. However, balancing single-track vehicles (STV's) such as bicycles and motorcycles, must be made to observe strict roll-angle bounds. Due to their unstable nature only closed-loop control can be of practical interest.

In the spirit of one learning to ride a bicycle our first concern will be balance. We reserve ground-path following and the associated problem of output tracking for nonlinear nonminimum phase systems for future work.

A feedback control law for a simple model of a riderless, powered two-wheel cycle will be derived. The control allows the vehicle to exponentially converge to and track smooth roll-angle and rear-wheel velocity trajectories. The simple bicycle model used here displays what might be considered to be the most important defining features of STV's; it is nonholonomic, nonlinear, and non-minimum phase. The control we will derive affords stable tracking of roll-angle, and non-zero rear-wheel velocity trajectories, affording the vehicle the ability to automatically recover from near-falls.

The derivation of the controller is through reduction of the nonlinear differential and algebraic equations of vehicle motion to a reduced state-space. Once the equations are reduced and a choice of desired trajectory is made, a nonlinear compensator is derived that input-state linearizes and exponentially stabilizes the error dynamics of the roll-angle and rear-wheel/ground contact velocity trajectories in the reduced state space. Simulations demonstrate the results.

2 Dynamics and Control

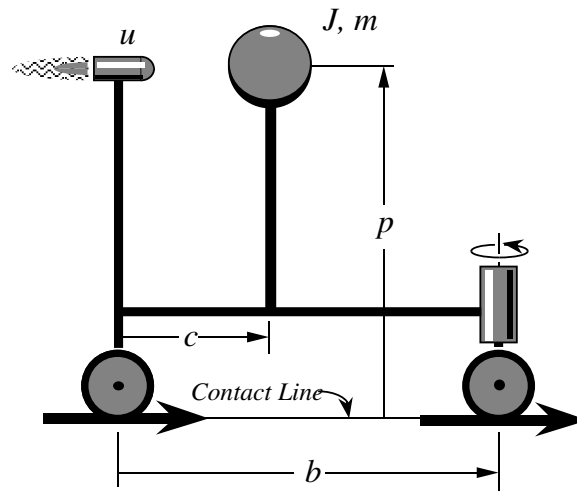


Figure 2.1: Side view of the bicycle model with $\alpha = 0$. Bold arrows indicate wheel-plane/ground intersection.

The control of a bicycle model illustrated in figures 2.1 and 2.2 will be considered. The wheels of the bicycle are considered to have negligible inertial moments, mass, radii, and width, and to roll without side or longitudinal-slip. The vehicle is assumed to have a fixed steering axis that is perpendicular to the flat ground when the bicycle is upright. An inertia tensor, J , is associated with

the bicycle frame and rigidly attached payload and is expressed in the center of mass frame of the vehicle. For simplicity it will be assumed to be a constant multiple of the 3 by 3 identity matrix, $J = jmI$. The rigid frame of the bicycle will be assumed to be symmetric about a plane containing the rear wheel.

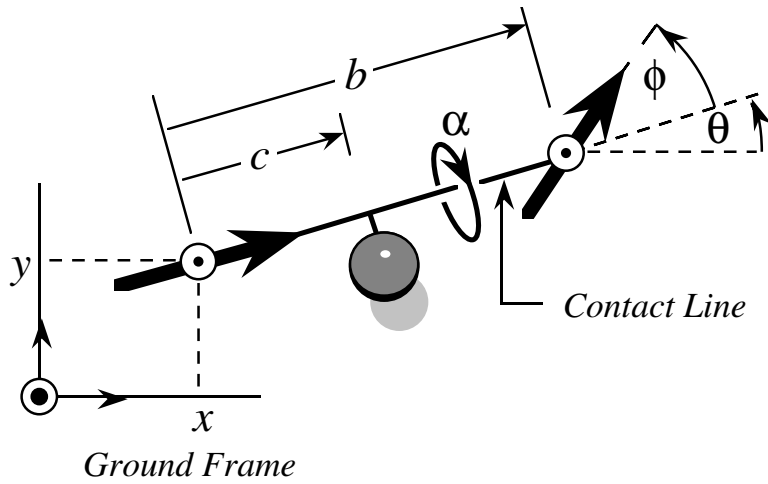


Figure 2.2: Top view of the bicycle model rolled away from upright by angle α . Bold arrows indicate wheel directions at the ground plane.

Consider a ground-fixed inertial reference frame with x and y axes in the ground plane and z -axis perpendicular to the ground plane in the direction opposite to gravity. The intersection of the vehicle's plane of symmetry with the ground plane forms a *contact-line*. The contact-line is rotated about the z -direction by a *yaw-angle*, θ . The contact-line is considered directed, with its positive direction from the rear to the front of the vehicle. Yaw-angle θ is zero when the contact-line is in the x direction. The angle that the bicycle's plane of symmetry makes with the ground is the *roll-angle*, $\alpha \in (-\pi/2, \pi/2)$. Front and rear-wheel contacts are constrained to have velocities parallel to the lines of intersection of their respective wheel planes and the ground-plane, but free to turn about an axis through the wheel/ground contact and parallel to the z -axis.

Let $\phi \in (-\pi/2, \pi/2)$ be the *steering-angle* between the front wheel plane, ground plane intersection and the contact-line as shown in figure 2.2. The generalized coordinates of the bicycle, assumed known, are $q := (\alpha, \theta, x, y)$. Variable $s := \tan(\phi)/b$, where b is the distance between the front and rear-wheel contacts, will be referred to as the *steering-variable* and its derivative, $w \equiv \dot{s}$ as the *steering input*. The bicycle is assumed to possess a rigidly attached force generator fixed in the vehicle's plane of symmetry and parallel to the contact-

line. For simplicity, and to assure ourselves that the angular momentum about a normal to the vehicle's plane of symmetry is always zero, it will be assumed that the force generator is directly behind the center of mass as indicated in figure 2.1. Thus the pitch of the vehicle may be ignored (If one wishes to consider the force generator to be at ground level, as with a tire, then $|\dot{v}_r|$ must be kept sufficiently small to avoid pitching the vehicle). The component of the force along the contact-line is the input u . We consider a vehicle with a rigid or non-existent passenger under automatic control.

Standard Lagrangian techniques lead to the bicycle model

$$\dot{s} = w \quad (2.1)$$

$$M(q_1) \ddot{q} = F(q_1, \dot{q}) + B(q_1)u + A^T(s, q_1)\lambda \quad (2.2)$$

$$A(s, q_1)\dot{q} = 0 \quad (2.3)$$

with $q_1 = [\alpha \ \theta]^T$, $q_2 = [x \ y]^T$, and

$$M(q_1) = m \begin{bmatrix} j + p^2 & -cpc_\alpha & pc_\alpha s_\theta & -pc_\alpha c_\theta \\ -cpc_\alpha & j + c^2 + p^2 s_\alpha^2 & pc_\theta s_\alpha - cs_\theta & ps_\alpha s_\theta + cc_\theta \\ pc_\alpha s_\theta & ps_\alpha c_\theta - cs_\theta & 1 & 0 \\ -pc_\alpha c_\theta & ps_\alpha s_\theta + cc_\theta & 0 & 1 \end{bmatrix}$$

$$F(q_1, \dot{q}) = m \begin{bmatrix} \frac{1}{2}p^2 s_{2\alpha} \dot{\theta}^2 + gps_\alpha \\ -cps_\alpha \dot{\alpha}^2 - p^2 s_{2\alpha} \dot{\alpha} \dot{\theta} \\ ps_\alpha s_\theta \dot{\alpha}^2 - 2pc_\alpha c_\theta \dot{\alpha} \dot{\theta} + (cc_\theta + ps_\alpha s_\theta) \dot{\theta}^2 \\ -ps_\alpha c_\theta \dot{\alpha}^2 - 2pc_\alpha s_\theta \dot{\alpha} \dot{\theta} + (cs_\theta - ps_\alpha c_\theta) \dot{\theta}^2 \end{bmatrix}$$

$$B(q_1) = [0 \ 0 \ c_\theta \ s_\theta]^T$$

$$A(s, q_1) = \begin{bmatrix} 0 & 0 & -s_\theta & c_\theta \\ 0 & c_\phi & -s_{\phi+\theta}/b & c_{\phi+\theta}/b \end{bmatrix}$$

where c , p , m , and j are constants in \mathbb{R} , and s_δ and c_δ refer to $\sin(\delta)$ and $\cos(\delta)$, respectively. Matrix M is positive definite and symmetric, and A is of full rank.

Equation (2.1) models the presence of a dynamically decoupled steering actuator. The steering-variable is *decoupled* from the vehicle dynamics, meaning that s is unaffected by q and \dot{q} when w is identically zero. Equation (2.2) models a vehicle subject to forces of constraint. Equation (2.3) models the nonholonomic constraints imposed by the front and rear-wheel contacts. The Lagrange multipliers $\lambda \in \mathbb{R}^2$ are states of the dynamical model that may be eliminated as will be shown.

Some important structural points of the bicycle model are: 1. The first and second entry of the matrix B are zeros indicating that there is no direct control of roll and yaw through u . 2. There is no explicit appearance of w in (2.2) indicating that the steering can only affect the vehicle behaviour through the *extrinsic* nonholonomic constraints [5]. 3. Suppose we choose initial conditions

$q(0)$ such that with particular inputs \bar{u} and \bar{w} the vehicle travels along a straight line in the plane at a constant speed. Then the resulting dynamics may be found by setting \ddot{x} , \ddot{y} , $\ddot{\theta}$, and $\dot{\theta}$ to zero. The resulting restricted *zero dynamics* is $\ddot{\alpha} = gps_\alpha/(j+p^2)$, the equation of an inverted pendulum, *unstable* about $\alpha = 0$. Thus, the bicycle is a *non-minimum phase* system [6] and, as experience would predict, it will fall over.

We follow the method presented in [5] with the addition of a linear coordinate change on the tangent bundle of the configuration space. Define a velocity transformation T and new velocity coordinates $\dot{\tilde{q}} = [\dot{\alpha}, v_r, \dot{\theta}, v_\perp]^T$

$$\dot{q} = T\dot{\tilde{q}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c_\theta & 0 & -s_\theta \\ 0 & s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ v_r \\ \dot{\theta} \\ v_\perp \end{bmatrix} \quad (2.4)$$

Transforming the constraint equations $A\dot{q} = 0$ to the new velocity coordinates yields a new constraint matrix $\tilde{A} := AT$. The new velocity variable v_r is the component of the velocity of the rear contact along the contact-line, and the velocity variable v_\perp is the component of the velocity of the rear-contact that is perpendicular to the contact-line. By virtue of the nonholonomic constraints on the wheels, $v_\perp \equiv 0$. The form of \tilde{A} suggests a natural partition of the velocity space. Let \tilde{A}_1 and \tilde{A}_2 be defined by

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\tan\phi \end{bmatrix} \quad \text{and} \quad \tilde{A}_2 = \begin{bmatrix} 0 & 1 \\ b & 1 \end{bmatrix} \quad (2.5)$$

so that $\tilde{A} = [\tilde{A}_1 \quad \tilde{A}_2]$. The corresponding velocity space partition is $\dot{\tilde{q}}_1 = [\dot{\alpha} \quad v_r]^T$, $\dot{\tilde{q}}_2 = [\dot{\theta} \quad v_\perp]^T$. The velocity coordinates $\dot{\tilde{q}}_1$ will be referred to as the *fibre* velocities, and the velocity coordinates $\dot{\tilde{q}}_2$ will be referred to as the *base* velocities.

We now form the *connection* map V that takes the base velocities $\dot{\tilde{q}}_1$ into the vector of all velocity variables $\dot{\tilde{q}}$.

$$V = \begin{bmatrix} I \\ \dots \\ -\tilde{A}_2^{-1}\tilde{A}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dots \\ 0 & s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} V_1 \\ \dots \\ V_2 \end{bmatrix} \quad (2.6)$$

The fibre velocities are connected to the generalized velocities by

$$\dot{\tilde{q}} = V\dot{\tilde{q}}_1 \quad (2.7)$$

Note that V_2 has rank 1 throughout the domain of interest $-\pi/2 < \alpha < \pi/2$. This implies that *different values of the fibre velocities may correspond to the same value of the base velocities*.

Differentiating (2.7) and (2.4) with respect to time and plugging the result into (2.4) yields

$$\ddot{q} = TV\ddot{q}_1 + (\dot{T}V + T\dot{V})\dot{q}_1 \quad (2.8)$$

Substituting (2.8) this into (2.2) and left-multiplying by $\tilde{V} := TV$ gives

$$\tilde{V}^T M \tilde{V} \ddot{q}_1 = -\tilde{V}^T \left(M \left(\dot{T}V + T\dot{V} \right) \dot{q}_1 - F \right) + \tilde{V}^T Bu \quad (2.9)$$

where we have used the fact that $\dot{A}V = 0$. Differentiation of V brings w explicitly into the dynamics. We may then divide both sides of (2.9) by m to obtain

$$\tilde{M} \ddot{q} = \tilde{F} + \tilde{B} \begin{bmatrix} w \\ u \end{bmatrix} \quad (2.10)$$

where

$$\tilde{M} \equiv \begin{bmatrix} j + p^2 & -cpsc_\alpha \\ -cpsc_\alpha & 1 + (j + c^2 + p^2s_\alpha^2) s^2 + 2ps_\alpha s \end{bmatrix}$$

$$\tilde{F} \equiv \begin{bmatrix} p(g s_\alpha + c_\alpha v_r^2 s + \frac{1}{2} p s_{2\alpha} v_r^2 s^2) \\ -ps\dot{\alpha}(c s_\alpha \dot{\alpha} + 2c_\alpha v_r + p s_{2\alpha} s v_r) \end{bmatrix}$$

$$\tilde{B} \equiv \begin{bmatrix} cpc_\alpha v_r & 0 \\ -((c^2 + j + p^2s_\alpha^2) s + ps_\alpha) v_r & 1/m \end{bmatrix}$$

Note that the structure of \tilde{B} is such that if $v_r = 0$, w is blocked from having any affect on α , confirming the experience that one cannot control a bicycle by steering alone when $v_r = 0$.

When $v_r \neq 0$ and \tilde{B} is invertible we may choose as input

$$\begin{bmatrix} w \\ u \end{bmatrix} = \tilde{B}^{-1} \left(-\tilde{F} + \tilde{M}v \right) \quad (2.11)$$

giving the input/state linearized reduced dynamics $\ddot{q} = v$. Let $\alpha_d(t)$ and $v_{rd}(t)$ be desired twice differentiable trajectories for roll-angle and rear-wheel velocity, respectively, with $v_{rd}(t)$ never zero. By choosing real numbers c_{11} , c_{10} and c_{20} such that the roots of the polynomials $r^2 + c_{11}r + c_{10}$ and $r + c_{20}$ are in C_-^o , we may set

$$v = \begin{bmatrix} \ddot{\alpha}_d - c_{11}(\dot{\alpha} - \dot{\alpha}_d) - c_{10}(\alpha - \alpha_d) \\ \dot{v}_{rd} - c_{20}(v_r - v_{rd}) \end{bmatrix} \quad (2.12)$$

to obtain a control law (2.11), (2.12). By such a control law we may obtain exponential convergence to the desired trajectories from any initial conditions such that $\alpha(0) \in (-\pi/2, \pi/2)$, and $v_r(0)$ has the same sign as $v_{rd}(t)$.

2.1 Simulations

Figures 2.3 through 2.8 show the results of two numerical simulations for a vehicle with parameters $p = 1/2$, $c = 1/2$, $b = 1$, $m = 20$, $j = 1/10$, and $g = 9.8$. The initial conditions for both simulations are $s(0) = 0$, $\alpha(0) = -\pi/4$, $\dot{\alpha}(0) = 0$, $v_r(0) = 2$, $x(0) = 0$, $y(0) = 0$, and $\theta(0) = 0$. The gains for both simulations were chosen to be $c_{11} = 2$, $c_{10} = 6$, and $c_{20} = 1$.

In the first simulation $v_{rd} = 4m/s$, and $\alpha_d = \pi/8$ radians. Figure 2.3 shows graphs of the resulting $\alpha(t)$, $v_r(t)$, as well as the resulting path in the plane. Figure 2.4 shows the inputs $w(t)$, $u(t)$, and the steering angle $\phi(t)$. Figure 2.5 shows the configuration of the bicycle for the first 7 seconds of simulation 1.

For the second simulation $\alpha_d(t) = \frac{\pi}{4} \sin(2\pi t/5)$, while $v_{rd}(t) = 5 + \sin(2\pi t/10)$. Figure 2.6 shows the convergence of v_r and α to their desired counterparts, as well as the resulting path in the plane. Figure 2.8 shows the configuration of the bike in the interval from 4 to 7 seconds.

3 Conclusions

A nonlinear controller has been derived that affords a balancing nonholonomic, non-minimum phase bicycle model the ability to track arbitrary smooth roll-angle and non-zero rear-wheel velocity trajectories from initial conditions well away from the desired trajectories. An important problem not yet solved is the derivation of a control that affords the vehicle the ability to track x, y trajectories in the plane while remaining upright. This is an example of the more general problem of output tracking for non-minimum phase systems, an area of active current research. It will be addressed by the author in future work.

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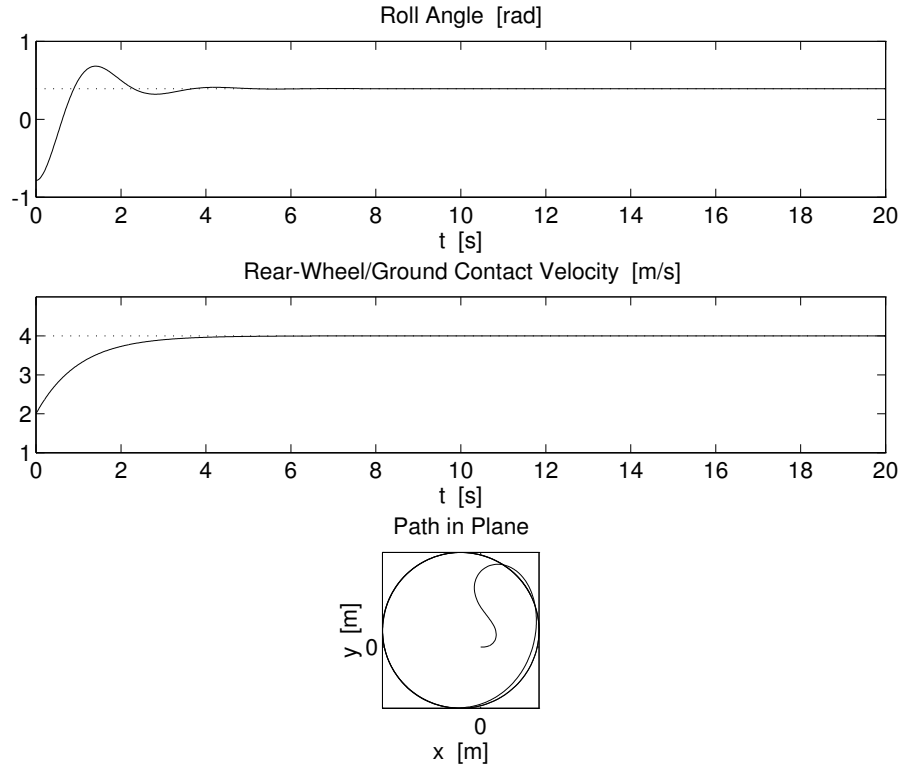


Figure 2.3: Simulation 1. Top: $\alpha(t)$ (solid) with $\alpha_d(t)$ (dashed). Middle: $v_r(t)$ (solid) with $v_{rd}(t)$ (dashed). Bottom: Path of the rear-wheel contact in the ground plane.

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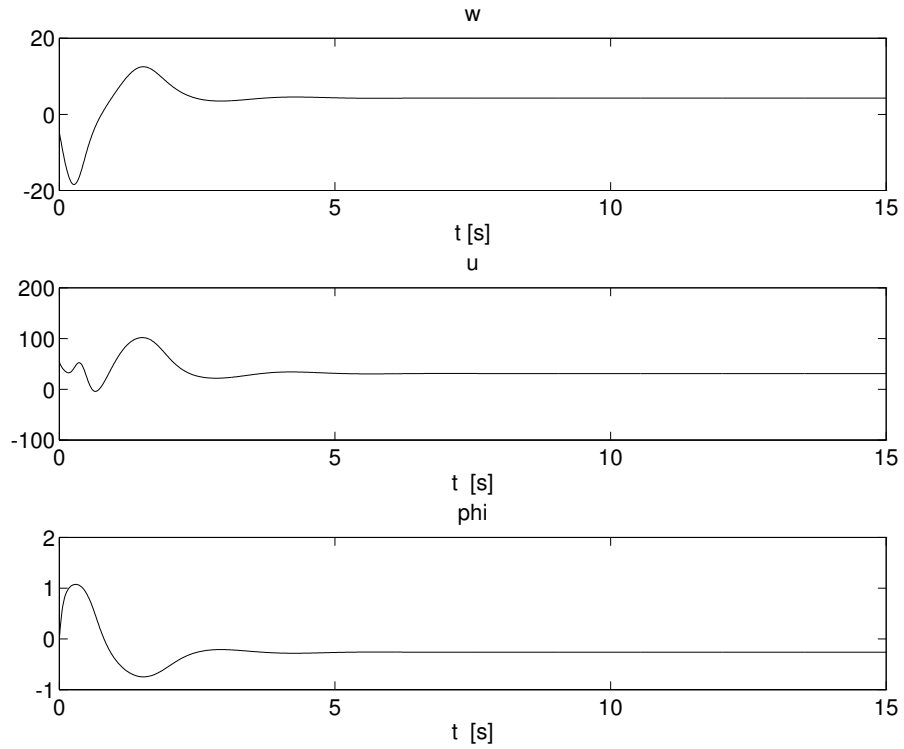


Figure 2.4: Simulation 1. Top: $w(t)$. Middle: $u(t)$. Bottom: $\phi(t) = \arctan(s(t)/b)$.

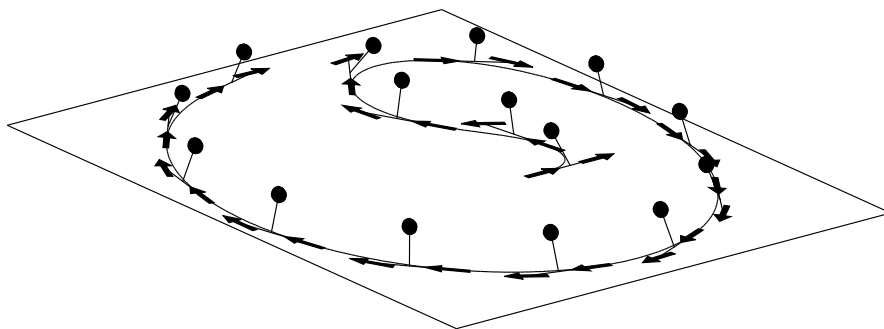


Figure 2.5: Simulation 1. The first 7 seconds of the ride showing the bicycle's configuration in 0.5 second increments starting from $t = 0$.

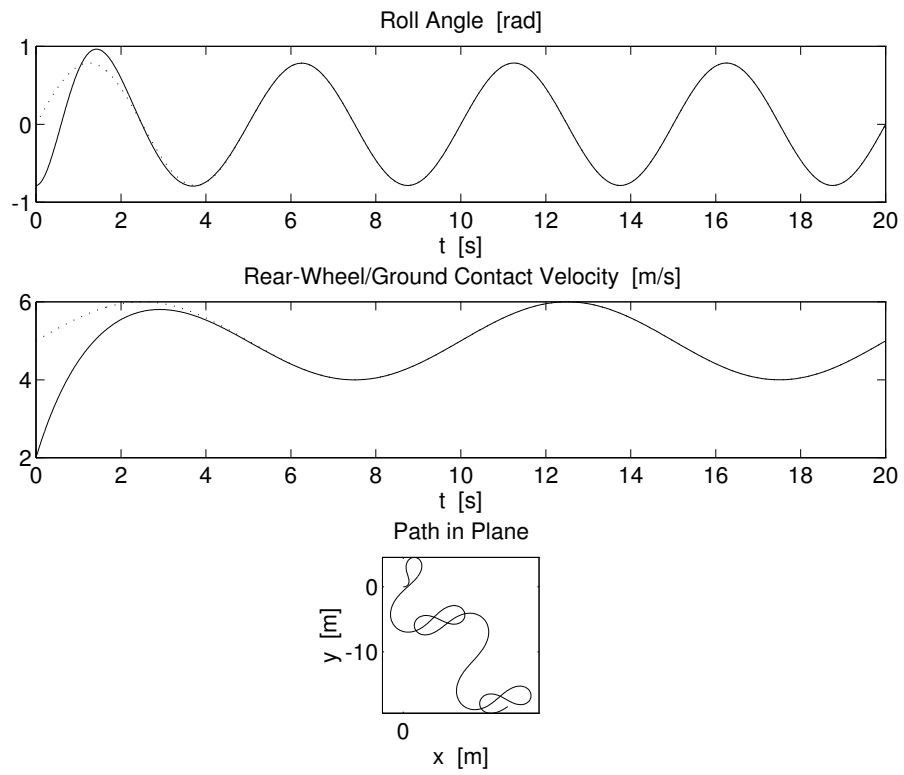


Figure 2.6: Simulation 2. $\alpha(t)$, $v_r(t)$, and ground plane path.

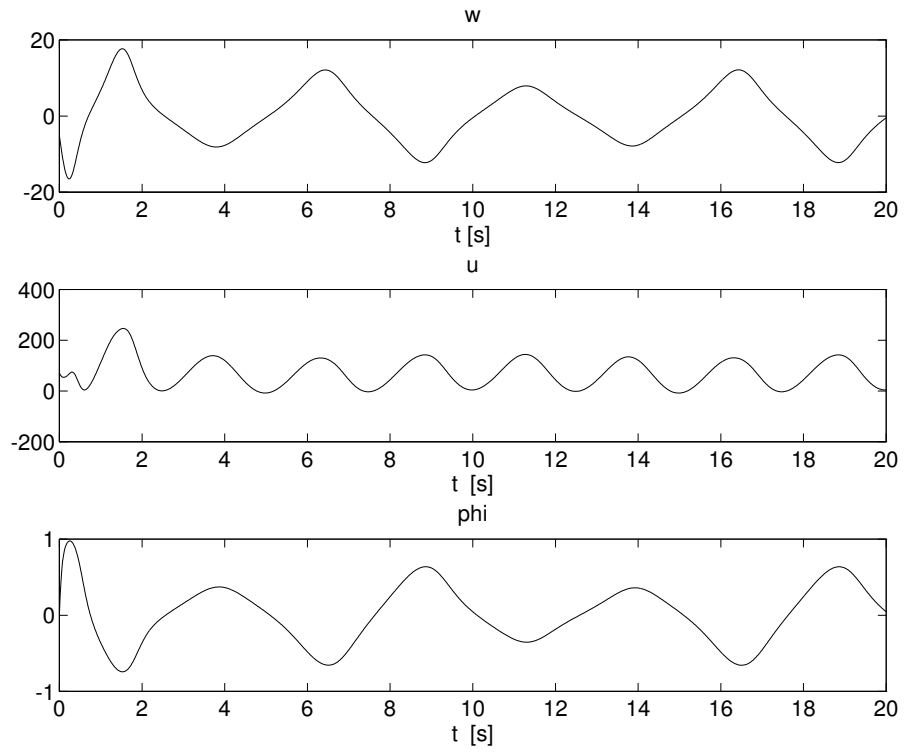


Figure 2.7: Simulation 2. Inputs $w(t)$ and $u(t)$, and steering angle $\phi(t)$.

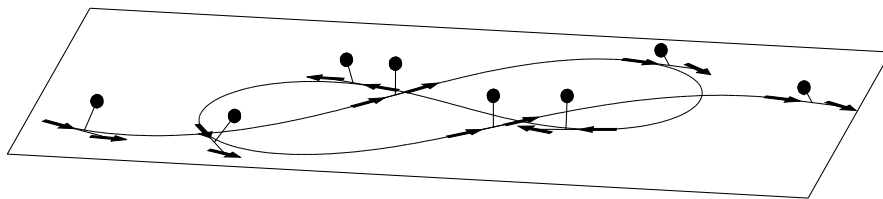


Figure 2.8: Simulation 2. Seconds 4 through 11 of the simulation 2 ride showing the bicycle's configuration in 1 second increments.